THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

62 COLERAINE ROAD, BLACKHEATH, LONDON, S.E. 3.

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CONTENTS.

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PREMATURE ABSTRACTION.*

BY W. C. FLETCHER.

I HAVE long been convinced that the teaching of mathematics suffers from the premature introduction of abstraction without adequate attention to the concrete facts from which the abstractions are derived. Perhaps I cannot better begin my argument than by taking as text a passage from the end of the first chapter of Todhunter's *Differential Calculus*:

"It often happens that a person commencing this subject is discouraged at the outset because he cannot discover any practical application of the abstruse points to which his attention is directed. It may therefore be of service to assure him that the difficulty is owing more to the nature of the subject than to his own want of comprehension.

The student must of course leave to his teacher the task of arranging and of selecting the definitions; he must at first be satisfied with reflecting on the meaning of the definitions and examining whether the deductions are correct. There are innumerable applications . . . but we shall at first confine ourselves merely to the logical exercise of tracing the consequences of certain definitions."

In other words, abstract science first, applications afterwards: pure mathematics followed by applied.

When a scholar sits down to write a textbook, such an attitude is natural, but a textbook is only fit for the use of those who either have some general acquaintance with the subject-matter or have already been so trained in abstract thought that they can take up a new branch of enquiry in abstract fashion from the outset. Boys and girls at school and for the most part students at universities also come into neither of these categories. For them a quite different introduction is required.

* [Mr. Fletcher has placed the Association still further in his debt by writing for the *Gazette* an article embodying the material which would have formed the basis of the Presidential Address which, had it not been for the war, he would have delivered at the 1940 Annual Meeting.—EDITOR.]

In the earliest stages the need is too obvious to be overlooked, and we do not introduce children to arithmetic without constant reference to familiar facts, nor to algebra by putting into their hands an abstract scientific treatise : " school books " of the elementary type are very different from textbooks. But in geometry the dominance of Euclid maintained the wrong tradition, and more generally the fact that university-trained teachers have learned most of their work in abstract fashion and forgotten their own early difficulties renders it natural that we should all be exposed to the danger of introducing our pupils to a new subject, or a new detail of a subject, without making sure that they have the background of concrete perception and knowledge which is the necessary presupposition for its abstract treatment.

I propose to examine, in the light of these general observations, first, some recent discussions and proposals in regard to geometry, second, some points in the teaching of arithmetic and algebra where I think it is useful to recognise that our familiar difficulties are in fact cases of the general difficulty of proceeding from the concrete to the abstract.

In two articles published in the *Gazette* (May 1924 and May 1927) Professor Baker gave us his views as to the desirable nature and content of school geometry. He raises questions which are still unsettled among us and some discussion of them will perhaps help me to bring out what I want to say. If my reference to them induces any of you to turn to them and study them with the care they deserve, that alone would justify the use to which I propose to put them.

His ultimate object, I think, is that when boys begin the Science of Geometry, as opposed to what he significantly calls its Natural History, they should do so with open minds, unprejudiced by fundamentalist assumptions. The science, in his view, should rest on the axioms of incidence and on one other ; for choice he takes Pappus' Theorem.

The Natural History should include not only, as it does now, properties for the mensuration of figures and parts of descriptive geometry which arise naturally as a development of geometrical drawing and model making, but also much more extensive study of the phenomena of incidence than we are accustomed to give.

In particular he puts forward three definite proposals—that propositions relating to middle points and propositions relating to perpendicular lines should be generalised, and that children should be practised in determining whether propositions depend on Pappus' theorem or not.

With his ultimate object I am not here greatly concerned. Only a very small percentage of our pupils will ever reach the study of the Science of Geometry, though I hope that better methods in school practice may increase this percentage. My interest is in the great mass, including both those who will pursue mathematics to the end of and beyond their school days and the still greater number for whom the subject will end at the School Certificate or earlier.

His expression the "natural history of geometry" seems to me to suggest most aptly the kind of work we should pursue at school, and to fit very exactly what I mean when I say that abstraction should be preceded by and gradually develop from the study of the concrete phenomena.

That Professor Baker is not solitary in his advocacy of "natural history" I may illustrate from the writings of other authorities of the highest eminence.

Hilbert in his preface to *Anschauliche Geometrie* says: "In Mathematics as in all scientific enquiry we meet two tendencies—to abstraction and to intuition. The former seeks to work out the logical points of view . . . and to bring these into systematic connection. The latter aims rather at a living grasp of the facts and the relations of their *contents*. . . .

To bring forward Geometry in broad features, using intuitional methods of exposition, should lead to a better estimation of Mathematics among wider circles of the public. For in general Mathematics, even if its importance is recognised, enjoys little affection. . . .

The reader will be led to walk in the great garden of geometry and everyone will be able to pluck a flower as he pleases."

This last sentence seems an echo from Felix Klein, *Die Entwickelung der Mathematik im* 19. *Jahrhundert*. Speaking of Salmon's books (I, 165), he says, "They are like refreshing and instructive walks through wood and field and cultivated gardens, where the guide draws attention now to this beauty, now to that strange appearance, without forcing everything into a rigid system of faultless perfection . . . and without digging out individual useful plants and transferring them to cultivated soil according to the principles of intensive cultivation. In this flower garden we have all grown up, here we have gathered the foundation of knowledge on which we have to build."

Poincaré again is possessed by the same kind of thought—see for instance "Les définitions mathématiques et l'enseignement" in *Science et Méthode*—but does not lend himself quite so easily to specific quotation.

That the natural history should include the phenomena of incidence I fully agree. These, as Professor Baker says, are much more manifest and important in solid than in plane geometry; in solid they arise inevitably, and as the tendency towards the early study of solids and the drawing involved therein develops, incidence will more and more force itself upon our notice.

I now turn to Professor Baker's definite proposals.

Two of these he presents as "generalisations". Generalisation is a natural aim for the developed mathematician, but it is not a process to which the ordinary boy takes kindly, being in fact a necessary aspect of abstraction, and I think that some method of procedure more likely to appeal to the immature mind of a boy is desirable; the generalisation should itself be suggested by experience.

As regards the harmonic property of the quadrilateral there is no difficulty. It is sufficient, for instance, to tell a class to take three points A, B, Q in line and perform the construction, dictated step by step, so determining the fourth point $P \equiv Q\{AB\}$. Then tell them to repeat the construction on the other side of the line. At once their interest is roused by finding that they arrive apparently at the same point P. Is this chance or is it inevitable, and if so, why? The proof that it is inevitable, or as I should prefer to say the explanation of the fact, from Desargues' theorem, is then appreciated.

That the general proposition so arrived at includes the fact that as Q recedes P approaches the mid-point of AB and reaches it when the ray from Q has become parallel to AB is easily appreciated. But something more is wanted, in my opinion, to connect the new truth with the concrete facts of common life. But as this appears more clearly in connection with perpendiculars I will defer its discussion.

Professor Baker himself is more doubtful about including this second generalisation in school practice. He does not give his reasons, but I will give mine.

To construct the generalisation of the perpendicular to a given line OA requires the assumption of two points, I, J, and the determination of the ray OB, the conjugate of OA with respect to OI, OJ. But if, the construction having been carried out, we ask what happens as OA rotates, we find that OB, instead of rotating with it, approaches OA by rotating in the opposite direction—a fact difficult to reconcile with our present ideas of perpendicularity.

Even for those who are ultimately going to pursue geometry in its higher developments surely some more satisfactory approach is desirable, and for those who are not, it seems to me essential. So I ask again, how can the new phenomenon be reached from concrete experience?

This is to ask the wider question, what is the field of experience from which the science of projective geometry did actually arise and from which it should still be made to arise if it is to interest any but a small body of specialists? I think the answer is: the study of the external world by vision alone when we can no longer rely on touch and direct measurement. Such study involves drawing in perspective, the full perspective of central projection, whereas up to this point axial projection has been sufficient.

I think then that, certainly for school boys, and in my opinion for university students also, scientific projective geometry should be preceded by or, in its early stages, combined with some practical study of "perspective" as pursued in a school of art.

The earliest stages of such study are certainly possible in school and almost at the outset the construction which gives the harmonic relationship forces itself on our attention and the equation of mid-points to harmonic conjugates.

Before we can reach the desired generalisation of perpendicularity,

however, something more is needed—the solution of the inverse problem: given a picture representing familiar objects, determine the position of the observer with regard to these objects. This inverse problem forms, it may be noted, part of the syllabus of the Board of Education for the elementary stage of its examinations in perspective. The suggestion that attention should be directed to it in school work is therefore not extravagant.

For my immediate purpose, this question of perpendicularity, I will assume that a picture is before us in which we can identify two pairs of lines known to represent pairs of perpendiculars and that the Horizontal Line of the picture has already been determined. If then these two pairs of lines meet the H.L. in P, Q and R, S respectively, the intersection of semicircles on PQ and RS determines the Centre of Vision and the Point of Station.

But if we set formal exercises on this it may happen, or be made to happen, that the semicircles do not intersect. The exercise presented an impossible state of things; things cannot be seen in the way suggested.

Yes, but as in other cases of "impossibility" (division, subtraction, evaluation of roots) mathematics is accustomed to persist, to carry on its regular processes in the expectation that sooner or later it will be led to new and valuable truth. So here we replace the common chord of two circles by the more general radical axis and so determine an "imaginary" point of station, but a real centre of vision. We are led also to two real points I, J on the H.L. with regard to which our pairs of lines alleged at the outset to be perpendiculars are harmonic conjugates.

We find also that we have to distinguish, in assigning lengths to segments of lines, between lines whose vanishing points lie within the segment IJ and those where they do not.

We are led in fact in a perfectly natural fashion from a concrete problem—the interpretation of a picture—to the basic phenomena of what Klein calls "Pseudo-euclidean" geometry—the geometry which lies at the base of the Special Theory of Relativity.

Incidentally I may say that in perspective perhaps there may be found means for reconciling the plain man or the ordinary boy to Professor Baker's view that "distance" is a difficult idea and to that of the first Geometry Report that "congruence" is an unsatisfactory basis for geometrical theory. In a perspective drawing, as in life when vision is the only sense we may use, distances and congruences are certainly not easy to recognise, and even if from other sources we have the ideas, we have to devise new and perhaps rather fantastic substitutes for the direct measurement with material instruments to which we are otherwise accustomed.

How far it may be possible to introduce the process I have been sketching into school work I am not prepared to say, but I feel that without some such preliminary boys would regard the generalisation of perpendicularity as barefaced sophistication—and that they would be right in so regarding it.

I turn now to Professor Baker's third detailed suggestion concerning Pappus' theorem and the use to which it should be put. First as to the theorem itself. Just as with the harmonic property of the quadrilateral, the fact of the collineation is easily discovered by drawing and excites surprise and delight—and the question why? Professor Baker regards it as absurd to prove the theorem from similar triangles, and from his advanced point of view no doubt he is right. But the question in the boy's mind is not so much is the apparent fact actually true, but *why* is the fact so ; he needs, that is to say, not so much to confirm the belief he has formed on approximate evidence, as to relate the newly accepted fact with his existing knowledge or beliefs. For him, therefore, the proof which Professor Baker discards is of real interest and value. Later on when he comes, if he ever does, to examine the whole of his geometrical knowledge and consider which elements of it he shall adopt as the axiomatic basis for the deduction of the rest he may well adopt Professor Baker's position and the fact that he formerly proved Pappus' theorem from congruence need be no hindrance to his doing so.

The fact on which Professor Baker lays so much stress, that Pappus' theorem cannot be derived from incidence alone can at least be rendered probable and reasonable to boys. The figure of Desargues' theorem (triangles in perspective) exemplifies a truth of plane geometry, but it can equally well be interpreted as a drawing of a three-dimensional figure, namely, a triangular pyramid cut by a plane. The ten points and ten lines of which the figure consists represent actual points and lines in the solid figure.

With the figure of Pappus' theorem it is not so. It can of course be regarded as representing things in three dimensions, say a post standing upright and another lying on the ground with six wires connecting three points on the one with three points on the other. But the crossings of these wires are not intersections ; they are merely accidental, depending for their position on the position of the spectator. There is no obvious reason why these three points should be in line. From this point of view, then, there is a radical difference between the two propositions and we can accept without repugnance the authoritative statement that Pappus' theorem cannot be deduced from propositions of incidence alone.

The last point in Professor Baker's specific suggestions is that boys should be taught to distinguish between propositions which can be proved without Pappus' theorem and those which cannot. This, I think, is probably a question for a somewhat remote future. All Euclid's propositions depend on congruence, and while we can ask, and do ask, the analogous question does a proposition also depend on parallelism or not, there would, while we are still in this limited region, be no point in asking whether they depend on congruence, which, if I understand the matter rightly, is equivalent to asking whether they depend on Pappus. So it will only be when we are at least beginning to breathe the freer air of projective

geometry that the question will have relevance. How soon this will be I am not prepared to say.

If I seem to have travelled unduly far from my main theme in discussing these proposals of Professor Baker's my defence would be that they seemed to afford a comparatively untrodden field for the exemplification of my general principle. His proposals have for myself a further attraction. At our Annual Meeting in 1939, when the Second Report on Geometry was under discussion, a speaker complained that it was not sufficiently revolutionary. Also, so far as I know, Professor Baker's proposals have never received at our hands the thorough detailed discussion they deserve. I have quite consciously been using the present occasion to outline the direction in which it seems to me that revolution might go, with what indications my limited experience suggests as to its possibilities, and above all to emphasise what I am putting forward as a cardinal principle, that every advance in school mathematics must be carefully based on familiarity with the concrete facts from which it arises.

As I have sometimes ventured to say: seventeenth century mathematics is better material for boys than that of the nineteenth and twentieth.

I pass now for a moment to the calculus, with reference to which the passage I have quoted from Todhunter was written. I suppose that in his days the subject was reached at school by very few. I do not think that this justifies his attitude in writing what was presumably meant as an introductory book even for college students. But I would ask you to contrast his treatment with that which, as I suppose and hope, is now becoming usual in schools.

We do not begin as Todhunter does with an abstract general definition of a differential coefficient, but with the examination of the gradient of a curve and of the rate of growth of the functions tabulated in our books of mathematical tables. This has an immediate though subsidiary consequence of great importance: our pupils find immediate occasion for their own activity instead of merely trying to understand and to absorb material put before them dogmatically. With but little guidance and at the cost, the very profitable cost, of simple arithmetic a boy can arrive at such a statement as: in the neighbourhood of 30° the sine of an angle is increasing at the rate of 0·0015 per degree. The mere qualitative study of graphs, the recognition that here the graph is rising steeply, there falling slowly, and so on, is itself of great value, preparing the mind for recognising the need for measurement of rates and suggesting the means by which the measurement can be effected. Similarly the old, Todhunterian, introduction of the integral needs, and I hope receives, suitable preparation—the actual calculation of what are in fact definite integrals, though they will not be so named till the process has become familiar.

Perhaps I may be permitted to recommend to teachers to whom it is not familiar the study of Mercator's Chart as a field in which

the basic principle of integration can conveniently be exercised. The chart is a familiar fact and it is easy to rouse curiosity as to how it is made, while the actual calculation of some of the intervals between parallels is an excellent example of the process of integration. Napier's method of calculating logarithms is another such field but not quite so easily cultivated as that of Mercator.

It seems to me that time would be far better spent on the careful study of such things as these—things of interest to the ordinary man, not merely to the mathematician—than on the refinements and tricks of which the older books largely consist. Certainly I think this is true for the great mass of our pupils and for myself I think it not less true for the few who are mathematicians by instinct.

The mention of Mercator's Chart (*not* projection be it noticed) leads me to suggest that much more attention might be given to the structure of other maps and to the simpler astronomical phenomena than our present practice allows. I recently set a small class of girls to plot from *Whitaker's Almanac* the course of the sun through the heavens; they took a three months period each, plotting the positions at 10-day intervals and were delighted to find how their several contributions fitted together to form a consistent and beautiful curve. They afterwards put in the positions of the moon during a month and so discovered how the moon follows the ecliptic and the relation of its period to that of the sun.

Possibly the new interest in solid geometry will lead us to bring into our mathematical work the elements of geometrical astronomy which used to form part of what was called physical geography. I know no region of thought in which intuitional insight into geometrical relations finds a better chance of awakening and developing. This would naturally lead to bringing the elements of spherical trigonometry into our work. In my Cambridge days spherical trigonometry was just a mass of complicated formulae with little relation to anything else, and it was not till years afterwards, when I became interested in Poincaré's model of the hyperbolic geometry, that the subject became, to me, really alive and useful.

All this is doubtless revolutionary and not easy of realisation while we are bound by our own antecedents, by tradition, and by examinations; but some such revolution is necessary if mathematics is to come to enjoy not merely the distant respect, but the affection which Hilbert desires for it.

Perhaps before passing on to more elementary work I may give an example of appeal to the concrete in a detail of work. In discussing the graph of $1/x$ my girls were as usual a little bothered by what I called the breaking of the curve. Why should a curve break? This led me to make them plot the outline of the patch of light thrown through a circular hole by a lamp standing behind the hole; first when the lamp was higher than the top of the hole, then lower. In the latter case the actual curve is incomplete, but can be completed mathematically. To enforce the point I gave them some abstract cases of loci resulting in hyperbolas.

I pass now to some remarks on more elementary work.

The traditional school arithmetic has been built up largely on material derived from two spheres of human activity—commercial transactions and " mensuration ", that is, the ascertainment of the quantities of material required to paper a room, to fill a tank, to build a wall.

Probably all calculations within these two fields can be conducted without going beyond addition, subtraction, multiplication and division of natural numbers. There will be ragged ends (if 5 buns cost 1s., ought one to cost 2d., $2\frac{1}{4}$d., $2\frac{1}{2}$d., $2\frac{3}{4}$d., or 3d.); calculations will be tedious, units will be small and numbers large; but there is nothing to necessitate mathematical development in the sense of generalisation or abstraction.

Within this range the teacher's difficulties are mainly that of establishing the necessary mastery of figures and that children have ordinarily so narrow a range of experience that the subject-matter is unreal to them. There is the further difficulty involved in children's limited powers of reading the ordinary, non-mathematical statement of a question. The consequence is that school arithmetic in its ordinary range does comparatively little to develop abstraction: for the most part it is not in the strict sense mathematics nor even " arithmetic " in the sense in which the thoroughgoing pure mathematician uses the word. It is in what we call algebra but which he commonly calls arithmetic that we get a little nearer to his point of view, and the elements in school arithmetic which do lead towards abstraction are perhaps best considered in connection with " algebra ".

This does not mean that school arithmetic is unimportant or that it should be hurried over and relegated as soon as possible to a subordinate position. On the contrary, if it is not mastered, both in its mechanical aspect, speed and accuracy in pure calculation, and in the wider aspect of penetrating to the nature of a given situation and determining what arithmetical processes are required, progress in mathematics proper will be constantly hampered. With this preliminary caution, let us turn to algebra, that is, to the more strictly mathematical form of arithmetic.

Mr. Siddons has sometimes quoted to us the opinion of Professor Cayley as to the proper way of beginning geometry, viz. that we should commence with n dimensions, then reduce to three and finally to two. I suspect myself that a similar mistake was made in Italy some years ago under the influence of a leading mathematician when it was ordained that Euclid having been found full of unrecognised assumptions, schools should adopt a much more rigorous system based on the most recent analysis of ultimate postulates. I even venture to think myself that there is more than a trace of this state of mind in some of the appendices to our recent Report on the Teaching of Algebra.

By Algebra in the following observations I mean approximately the range of work for the School Certificate or rather less, and have

particularly in view that part of the work which to-day is rather under a cloud—contemptuously referred to as a " box of tricks ".

We all recognise that an essential, and perhaps the essential, step is the development of the power of formulation of processes already familiar in arithmetic. In days earlier than most of us can remember I believe that " arithmetic " consisted largely in learning formulae by heart and using them. My own school experience began about the time when this method was passing away, and to do " interest " sums, for instance, by formulae was condemned as unintelligent. To-day we deliberately reverse the process; teach " arithmetic " in what we think an intelligent fashion and then set to work as the beginning of algebra to formulate the processes we have been using. We recognise further that this latter process can with advantage be associated at a quite early stage with the teaching and practice of arithmetic itself, so that the two subjects are closley associated and help one another.

For instance, in dealing with the first step in " reduction to unity " it is useful to ask, repeatedly and at intervals, such questions as: if a things cost x shillings, what does one cost; if b men take y hours to finish a job, what will one take?

The primary advantage of this is to emphasise the nature of the process to be adopted, division or multiplication as the case may be. But there is a secondary advantage of no small importance. A boy in thinking about the question will think in terms of integers and of small, suitable integers; but if he does this and comes to the right decision—again and again so that the choice becomes automatic—it becomes easier for him to make the right decision when the numbers, being large or complicated with fractions, elude his grasp of the concrete. The transition from " if 2 things cost 6d." to " if $2\frac{1}{2}$ yds. cost 5s. 3d." becomes easier.

The introduction of fractions in this connection is particularly important, for while in the original statement ($2\frac{1}{2}$ yds.) the fraction is concrete and intelligible, when it is used as a divisor it becomes an abstract number and we are in fact led naturally and at first unconsciously to the first step in the extension of the number concept.

The most difficult step of all in this connection occurs when we have such questions as: if $\frac{3}{4}$ of a yard costs . . . , or worse still: if $\frac{1}{4}$ of a yard costs . . . . We can of course ride off on another and in a sense more natural way of dealing with these questions, viz. in the latter case deciding that a whole yard costs 4 times as much, but we have not got to the bottom of the matter till it is felt that division by the " number " of yards is always in place or till the shock of finding that " multiplication " by a fraction may make the result less and division may make it greater has passed away—not until in fact the mind has become reconciled, subconsciously perhaps but really, to this particular extension of the number concept. That this process is facilitated by the general algebraic formulation seems indisputable.

Perhaps I may enforce the point by reference to a criticism some-

times passed on "reduction to unity" to the effect that the second line is often absurd and should therefore be suppressed. I do not agree with the conclusion. The essence of the process lies in the decision to divide or multiply as the case may be, not, as in the earliest stages of the work, in the appreciation of the concrete facts. For the same reason, among others, the result should not be worked out in the second line, but left standing in the form x/a or by, as the case may be.

In many arithmetical questions, of the miscellaneous kind called problems, it is often advantageous to use algebra and reduce the problem to an equation. Is there anything in this that bears on the general matter which I am discussing? Yes, I think there is. The fact that it is useful to introduce an x arises from the fact that the problem is a backwards-way question: the final conditions are known, not the initial. Hence straightforward arithmetic cannot bite upon the data.

All such questions, as we know only too well, present difficulties to the slower minds. "A man buys a house for £800 and sells it at a profit of 10 per cent.; what did he get for it?" is straightforward and easy; the inverse question: "a man sells a house for £1000, making a profit of 10 per cent.; what did it cost him?" is difficult just because we cannot operate immediately on the £1000, but are driven in effect to abstraction—"if the original cost was £100, or £x, . . .".

In my school days when it was inadmissible to use algebra in an arithmetic paper—or the calculus in the "First Three Days"—the "x" was forbidden, though possibly S or P might scrape through. Now when barriers are less rigid it seems better to use algebra frankly when it is available and useful, not merely to facilitate the arithmetic, but as a valuable application of a subject which finds only too little practical scope in its early stages.

The algebraic formulation we have considered so far—such as arises naturally out of arithmetic and such as can be added to this out of concrete material within boys' experience—is all very simple in character and gives little opportunity for exercise in transformation of algebraic expressions of any but those of the simplest character. If we attempt to transcend this limitation by the quotation of, *e.g.* "formulae from an Engineer's Notebook", we pass into a field outside boys' concrete experience and the exercises become as purely formal as those in the most antiquated school books. The question must be faced: what is the justification for spending school time on practice in the merely formal side of algebra—factors, fractions, elaborate equations, identities?

For anyone who is going to pursue mathematics for its own sake the necessity is obvious, but this applies only to a minute percentage of our pupils; what about the great majority? Are we just dragging them along in the supposed interest of the very few, or because these few cannot at first be identified and we must cast our net broadly to be sure of catching them?

These are questions not easy to answer in regard to any school subjects and we must not be discouraged if we cannot find a plain answer to a plain question.

For myself I can only answer it or satisfy my own mind by considering my own teaching experience. I am concerned with a handful of girls, none of whom will ever be mathematicians ; some get as far as School Certificate, some not as far as that. They are handed over to me with a free hand to do what I think fit—having regard to the requirements of the School Certificate for some of them.

At first there is no difficulty ; algebra comes in, as I have already indicated, to help the arithmetic ; but am I justified, apart from the School Certificate, in pursuing the subject beyond these limits? I would rather turn the question into the form : how shall I handle the subject as to get the maximum of value out of the fragments which are all that in any case can be covered?

The range of work specified above may be included in the one word manipulation, and the essence of the process lies in learning to follow the rules which have emerged from arithmetic when there is nothing but the rules to rely upon and when appearances suggest, if they suggest anything at all, neglect and infraction of the rules. By degrees, as progress is made, the eye develops a new sensitiveness ; if progress goes far enough a new sense of form and symmetry, of homogeneity will be awakened. But these require time and experience ; they cannot be hurried, and many of our pupils will never attain them. That which is essential for all, and which is also the basis of experience from which the fuller acquisitions must be developed, is just faithfulness to rule, the patient and accurate performance of mechanical processes. Elegance and brevity may come in time and should of course be exhibited in any work the teacher does himself ; but they cannot be expected and ought not to be demanded in boys' own efforts till by experience and practice, faithfulness and accuracy, however clumsy, have been developed.

For this purpose I find the following sort of procedure useful, both as giving the necessary practice in manipulation and as keeping in contact with the relatively concrete and appreciable arithmetic.

The rules, of addition, subtraction, multiplication and division are a formulation in more abstract form of the familiar facts of arithmetic. Simple formulations again are abstract expressions of familiar experience. Equations of the simplest kind, their solution and verification, arise in the same way. But a time soon comes when, if manipulation is really to be practised, we must go beyond this simple material.

The best field, I think, is afforded by more elaborate equations, because these afford the necessary practice and also admit the concrete test of verification. Just because the various "rules" come in in a miscellaneous fashion and there is constant and miscellaneous practice in the abbreviations of algebra, work on equations is, I think, more profitable than separate practice, *e.g.* on factors, fractions, evaluations, and so on ; though when any

particular difficulty arises, a few *ad hoc* examples may well be set to overcome it.

I may illustrate one point which seems to me of great importance by reference to a book which I came across in use at a school. The earlier part of it was carefully restricted to signless numbers; but I found that when I asked for verification of the solutions to equations, negative numbers inevitably presented themselves, just as fractions would have done had the examples been constructed entirely in terms of integers. It is, I think, only through personal experience of the fact that the "rules", faithfully followed, work out sensibly, that real confidence in the subject is gained.

It is important, in my view, to carry the practice of equations to those with literal coefficients, that is, to those in which the abstract character is most fully developed. It is only at this point that the "rules" and the language of algebra reach their full development. "Verifications" afford practice in manipulation of a particularly valuable kind, more interesting, because they arise out of other work and with a definite object, than similar examples set arbitrarily.

That formal work of this kind should be associated with other work which shows directly possible concrete meanings for fractional and negative numbers ought to go without saying. Practice in drawing graphs is invaluable, and from time to time there should be short discussions of the meaning of addition, subtraction and multiplication as applied to relative or directed numbers. But I think myself that this work should accompany and follow formal practice rather than precede it. The transition from the familiar natural numbers of counting to the far wider and more abstract numbers of algebra is a very great step and requires variety of experience and practice and therefore much time for its accomplishment. As Klein says (*Elementar*, p. 30), "The thing cannot be precipitated, but for the revolution in his thinking, which completes itself in him through this knowledge, we must allow the pupil time."

The greatest hindrance to sound progress is hurry. This is true of many subjects, but most emphatically of mathematics, for here, in the early stages at least, we are concerned not mainly with the acquisition of predigested knowledge, but with the process of digestion itself—and the bolting of food meant for the mind can be as mischievous as the bolting of that for the body. The food offered for digestion must be suited to the recipient—milk for babes, strong meat for men; growth cannot be forced without risk of permanent injury and, to speak in terms of my subject, premature insistence on abstractions, generalisations, refinements, involves many of our pupils in disaster.

My own recent teaching experience has convinced me of what I long suspected that if the pace is adjusted to the individual there are few, if any, who cannot follow the early stages of mathematical teaching without being reduced to the state of disgust or despair which has been only too common. W. C. F.

AN INVESTIGATION INTO MULTIPLICATION.* I.

THE MULTIPLICATION OF DECIMALS.

By H. Webb.

Introduction.

There is probably no other item of the arithmetic syllabus which is confronted by such a diversity of earnestly supported method as is the case with the multiplication of decimals.

The problems which form the basis of this investigation are :

(*a*) What are the special advantages and disadvantages which bring about the adoption or rejection of these methods?

(*b*) What is the extent of these advantages and disadvantages?

(*c*) And finally a consideration of the methods of overcoming the disadvantages.

In a paper † read to the London Branch of the Mathematical Association on 16th March, 1929, Mr. R. A. M. Kearney described a new (or possibly very old) method of arrangement in the multiplication of decimals. In his opening remarks Mr. Kearney said, " It is perhaps somewhat remarkable that there should be no settling down by teachers of arithmetic into one standardised method and arrangement of work, shown by experience to be the simplest, safest, least laborious, and most easily understood and assimilated by the pupil." Long experience amongst my colleagues has taught me that we teachers are a most conservative body, particularly in the matter of changing of methods. Furthermore, there is a great deal of difference between the needs of the varying types of schools, and each type is, perhaps unwisely, more ardently concerned with its own immediate needs. An example of this is that the senior schools are not, as a general rule, concerned with contracted methods of multiplication in decimals. Mr. Kearney's method, as will be seen from the examples on pp. 87-88, demands a certain amount of co-operation on the part of junior and preparatory schools when these schools deal with the multiplication of whole numbers. He claims for his method that :

(*a*) it can be applied without break from the early stages in the teaching of arithmetic ;

(*b*) the arrangement is the same for whole numbers or decimals ;

(*c*) the position of the decimal point is fixed automatically by rule ;

* I should like to take this opportunity of thanking Mrs. Linfoot, Secretary of the Bristol Branch of the Mathematical Association, for much helpful advice and criticism during the progress of this work.

† See the *Mathematical Gazette*, May 1930, pp. 103-110. It may be a help to other students to know that I was able to see these back copies of the *Mathematical Gazette* through the courtesy of the National Central Library who obtained them for me from the Science Museum, South Kensington.

(*d*) standard form is dispensed with ;

(*e*) the rules for contraction are simple and obvious once they are pointed out.

Although this method does away with standard form Mr. Kearney asserts that pupils should learn it at some stage in their study. Quite rightly he says: "This will enable the pupil to understand without difficulty the sometimes puzzling subject of 'characteristics'.

Thus $$92{,}800{,}000 = 9{\cdot}28 \times 10^7$$
and $$0{\cdot}000572 = 5{\cdot}72 \times 10^{-4},$$
whence it is obvious that the characteristics of these logarithms are respectively 7 and -4.

The student will also be prepared for the method universally adopted in Physics for expressing large and small numbers."

Apart from his own, Mr. Kearney goes on to suggest that an alternative method for the multiplication of decimals is to put both of the factors in standard form thus :

$$78{\cdot}06 \times 0{\cdot}0003427.$$

$$\begin{array}{rl}
7{\cdot}806 & \times 10^{1} \\
\underline{3{\cdot}427} & \times 10^{-4} \\
23{\cdot}418 & \\
3{\cdot}1224 & \\
{\cdot}15612 & \\
\underline{54642} & \\
\underline{26{\cdot}751162} & \times 10^{-3} \\
= \underline{\underline{0{\cdot}026751162}} &
\end{array}$$

I must agree that such a form would make an excellent link with the work on logarithms, and from my experiences, particularly as a result of this investigation, it would be the neatest arrangement and give reasonably good results. In a senior school, however, such work on logarithmic characteristics would, normally, have to be postponed until some time after the multiplication of decimals had been dealt with.

MR. KEARNEY'S METHOD.

A. With the Multiplication of Whole Numbers.

(i)	(ii)	(iii)	(iv)	(v)
329×4	329×40	329×43	3724×541	2126×302
329	329	329	3724	2126
4	40	43	541	302
1316	13160	13160	18620	63780
		987	14896	4252
		14147	3724	642052
			2014684	

B. With the Multiplication of Decimals.

(vi)	(vii)	(viii)
329·71 × 43·08	786·2 × 0·000127	0·00683 × 572·1
329·7 1	786·2	0·00 683
4 3·08	0·0001 27	5 72·1
1318 8·4	0·0786 2	3·415
98 9·130	157 24	4781
2 6·3768	55 034	1366
1420 3·9068	0·0998 474	683
		3·907443

Rule. Place the multiplier with its first significant figure (that is, the first figure other than a nought) under the units figure of the multiplicand. The first partial product is placed with its figures under the corresponding figures of the multiplicand and the decimal point is under that in the multiplier.

Mr. Kearney's method is described in the *Teaching of Arithmetic in Schools* (a Report prepared for the Mathematical Association), Bell (1937), but I have not yet come across a school which has adopted it. For this reason I have not included it in my Final Comparative Tests. It does seem to me, however, that the method would suffer tremendously from bad alignment owing to the two positions of the decimal points (cf. also Method C, Final Comparative Tests). An idea of the difficulties of spatial arrangement created by the two positions of the decimal points may be obtained from a glance at examples (vi), (vii), and (viii) above. Note the spaces it was necessary to leave beneath the decimal point in the multiplicand. Mr. Kearney himself admits that he had no opportunity of testing the efficiency of the method and adds a note of warning that " care must be taken to place all figures correctly in columns ".

In the July (1930) number of the *Mathematical Gazette* there appeared an article entitled, " Decimal Processes : ' Tracking the Unit '," by F. C. Boon. This method is described and tested in the later stages of this investigation. Mr. Boon considers that, amongst other advantages, this method is mechanically smooth in operation and free from pitfalls, and that it depends upon and is a reminder of a fundamental principle. By means of a series of well-chosen illustrative examples Mr. Boon endeavours to interest teachers in " at any rate giving the method a wider trial ".

Dealing only with multiplication commencing with the left-hand figure of the multiplier and with the multiplier in a position beneath the multiplicand there are, at least, four methods worthy of careful consideration. From enquiries * I have discovered that these four methods were the most popular. The following paragraphs contain examples and a preliminary review of each one of them.

* A list of schools, and an indication of the method of multiplication of decimals adopted by them, is given at the end of this article.

A Review of Those Methods Most Commonly Used in the Multiplication of Decimals.

Method (i). *The Fractional Method.*

Example : 42·613 × 21·3.

$$
\begin{array}{r}
42613 \\
213 \\
\hline
85226 \\
42613 \\
127839 \\
\hline
9076569 \\
\hline
\\
\underline{\underline{907{\cdot}6569}}\ \ \textit{Ans.}
\end{array}
$$

In this method the two factors are multiplied together as if they were whole numbers. The position of the decimal point in the final product is determined by dividing by the requisite power of 10 obtained by counting the total of the decimal places in the factors.

This method is the most popular in elementary schools and, in spite of the fact that it is often dubbed "illogical" and "unmathematical", finds favour in quite a number of secondary schools. It gives, undeniably, the best results. Why then is it not universally adopted and why should it be labelled unmathematical? In the first place the adjectives "illogical" and "unmathematical" may not be justified. They were used, I discovered during my enquiries, mainly by teachers who did not favour the fractional method. The justification claimed for this apparent slander lies in the facts that:

(*a*) the method requires no thought in the matter of arranging the factors and, subsequently, in positioning the partial products;

(*b*) the method is not derived directly from the decimal notation but indirectly through vulgar fractions;

(*c*) the pupil carries out his multiplication according to first principles and then automatically counts places to determine the position of the decimal point. This indicates that the pupil has no intelligent interest in, and understanding of, the operations he is performing.

Surely the criterion of the success of any method lies in the fact that eventually it becomes so automatic that its demands upon the reasoning powers become as small as possible. Thus more mental energy is set free for dealing with other matters. It seems to me, too, that the pupil will comprehend this method at least as intelligently as he will any of the others. This, of course, depends largely on the method of approach adopted by the teacher. I myself recommend that this method be approached through the following stages:

(i) $\cdot 4 \times \cdot 3 = 4/10 \times 3/10 = 12/100 = \underline{\cdot 12}$.

(ii) $1\cdot 1 \times \cdot 09 = 11/10 \times 9/100 = 99/1000 = \underline{\cdot 099}$.

(iii) $(\cdot 03)^3 = 3/100 \times 3/100 \times 3/100 = 27/1000000$
$= \underline{\cdot 000027}$.

(iv) $23\cdot 612 \times 42\cdot 1 = 23612/1000 \times 421/10 = 9940652/10000$
$= \underline{994\cdot 0652}$.

From the above examples the rules governing the fractional method of multiplying decimals are made clear to the average senior school pupil. In fact a knowledge of this method becomes essential to all pupils when working examples similar to those in (i), (ii), and (iii) above.

The second objection to the universal adoption of the fractional method is that it does not lend itself to contracted work. This is true enough but I very much doubt whether, in these days of mathematical devices, the gain obtained by using contracted forms is worth the sacrifices necessary to make these forms reasonably efficient.

All the following methods lend themselves to contracted work.

METHOD (ii). *Example* : $42\cdot 613 \times 21\cdot 3$.

```
 42·613
    21·3
---------
852·26
 42·613
 12·783 9
---------
907·656 9
```

In this method the units figure of the multiplier is placed beneath the right-hand figure of the multiplicand.

The advantage of this method is the easy positioning of the first figures of the partial products. Their positions are fixed, as in the multiplication of whole numbers, by the position of their multiplying figures.

The disadvantages of this method are :

(1) The two positions of the decimal points often cause bad alignment. This is shown by the examples given here:

Example (*a*) : $81\cdot 65 \times \cdot 417$.

```
81·65
   0·417
---------
3·2660
   8165
   57155
---------
3·404805
```

Example (*b*) : $62\cdot 4 \times 21\cdot 7$.

```
 62·4
    21·7
--------
124·8
   6·24
    ·4368
--------
131·4768
```

Both of these examples were taken from pupils' test papers. In example (*a*) the decimal point of the multiplier has caused all the products to be 10 times too small. In example (*b*) the first and second partial products are 10 times, and the third partial product 100 times, too small. In all cases the errors are due to the space occupied by the decimal points.

(2) In examples such as ·06732 × ·312 the empty space between the multiplicand and the first partial product often results in bad alignment between these two quantities.

```
·06732
    ·312
--------
·020196
   6732
   13464
--------
·02100384
```

METHOD (iii). *Example* : 42·613 × 21·3.

```
 42·613
 21·3
-------
852·26
 42·613
 12·7839
--------
907·6569
```

As all the units figures are directly beneath each other this method is often referred to as " Tracking the Unit Method ". This method possesses the advantages that there is no preliminary arrangement of the decimal points, and owing to all decimal points being in line the bad alignment in Method (ii), examples (*a*) and (*b*), is entirely done away with.

In this method the first figure of a partial product is the same number of places to the left or right of the right-hand figure of the multiplicand as the multiplying figure is to the left or right of the units figure of the multiplier.

This rule creates a difficulty for the pupil, particularly in the early stages, as it is necessary for him to find a fresh position for the first figure of the first partial product for every different example.

A further difficulty is created with examples such as :

·3127 × ·00054.

```
·3127
·00054
----------
·00015635
·000012508
----------
·000168858
```

Here the first figure of the first partial product has to be placed

well to the right of the example. Thus there is no proper guide to alignment.

METHOD (iv). *Standard Form.*

Example : $42{\cdot}613 \times 21{\cdot}3$.

$$\begin{array}{r} 426{\cdot}13 \\ 2{\cdot}13 \\ \hline 852{\cdot}26 \\ 42{\cdot}613 \\ 12{\cdot}7839 \\ \hline 907{\cdot}6569 \\ \hline \end{array}$$

This method consists of converting the multiplier to standard form (*i.e.* a number lying between 1 and 10) and then proceeding as in Method (iii). If, in converting the multiplier to standard form, the decimal point is moved to the left or to the right then the multiplicand must be compensated by moving its decimal point to the right or the left.

This method gets rid of all the disadvantages of bad alignment which exists with Methods (ii) and (iii). It also has the advantages of :

(*a*) easy positioning (compared with Method (iii)) of the partial products as the first figure of the first partial product is always in vertical alignment with the right-hand figure of the multiplicand, and of

(*b*) providing a ready means of obtaining a rough estimate of the answer.

Its disadvantage lies in the confusion caused by the initial juggling with the positions of the decimal points.

A PRELIMINARY INVESTIGATION.

The Tests.

In May 1938 I tested four classes of pupils with Methods (ii) and (iii) in the following manner :

(1) The best and worst of the four classes were shown how to work Method (ii) and were then given a test on it. The intermediate classes had similar instruction and a test with Method (iii).

(2) Five days later the position was reversed. The intermediate classes were instructed and tested in Method (ii) and the best and worst classes in Method (iii).

The Results.

The results of these tests are given below. They show that Method (ii) is quicker and gives a far greater degree of accuracy than Method (iii), *but* also that Method (ii) suffers from bad alignment (owing to the spatial arrangement of the decimal point

positions) very much more than Method (iii). Method (iii), (Tracking the Unit), however, showed a distressing number of errors owing to the incorrect positioning of the decimal point.

N.B.—1. Mechanical errors such as multiplication and addition errors were not counted in these preliminary tests.

2. The highest degree of accuracy obtained in any single test was 65 per cent.

Item	Method (ii)	Method (iii)
No. of papers - - - -	103	103
No. of examples - - -	515	515
No. correct - - - -	262	204
Average % correct - - -	50·9%	39·6%
Errors of decimal point due to bad alignment - - -	60	3
Other errors due to incorrect position of decimal point - -	80	254
Total time taken - - -	1794 min.	1891 min.
Average time per paper - -	17·4 min.	18·4 min.

Supplementary Work on the above Preliminary Investigation.

The poor results obtained with Method (iii) were directly due to the fact that the position of the first figure of the first partial product varied from example to example. On the other hand, the better results obtained with Method (ii) could be attributed to the fact that the partial products were positioned in the same manner as in the multiplication of whole numbers. It was this fact that made me feel that Method (ii) was one well worth further consideration. The question was, how could the bad alignment due to the spatial arrangement of the decimal points be overcome? I tried to do this in two ways. On the first occasion I tried the method of writing down the multiplicand in its proper form and then placing the multiplier beneath in the usual manner *but* without its decimal point; *e.g.*

$$42{\cdot}613 \times 21{\cdot}3.$$

```
  42·613
     213
 -------
 852·26
  42·613
  12·7839
 --------
 907·6569
```

When using this form I found that not only was alignment improved but a further advantage was derived from it. In working

Method (ii) during the preliminary tests a number of pupils had used the decimal point of the multiplier as an alignment for fixing the decimal position in the final product. With this form such mistakes were impossible as there was only one decimal point to follow.

I tested one class with this form and found an improvement over Method (ii) in the preliminary tests—71 per cent. accuracy against 65 per cent.

This modified form of Method (ii), however, failed in one respect. It failed to get rid of the bad alignment due to the spatial arrangement of the decimal point in the multiplicand. It still exhibited the same fault as is shown in the first partial product of example (*b*) on page 90. In an endeavour to remove this fault I introduced a further modification to Method (ii). This consisted of writing down the multiplicand as if it were a whole number and then writing down the multiplier as a whole number beneath it in such a manner that the original units figure of the multiplier was beneath the right-hand figure of the multiplicand. Thus in the example—$42{\cdot}613 \times 21{\cdot}3$, the first step is:

$$\begin{array}{r} 42613 \\ 213 \\ \hline \end{array}$$

The second step is to draw a vertical line between the 2 and the 6 of the multiplicand in the following manner:

$$\begin{array}{r|r} 42 & 613 \\ & 213 \\ \hline & \\ & \\ \hline & \\ \hline \end{array}$$

The vertical line then gives the position of the decimal point.

I tested two of the classes, originally tested during the preliminary tests, with this new modification and found there was a slight improvement. In this case it was 51·4 per cent. against 44·7 per cent. obtained in the preliminary tests with Method (ii). *But* as this test on the new modification was given some two months later I could not place a great deal of reliance on the results. Later, during the final tests (see following pages), I discovered that the vertical line, representing the final decimal position, is more likely to cause bad alignment than the decimal point itself. This is no doubt due to the line being unaccommodating, rigid, and inflexible, whereas the decimal point position is always capable of being slightly adjusted. This point raises a possible objection to the use of vertical lines when working contracted forms of multiplication in decimals.

FINAL COMPARATIVE TESTS ON THE METHODS OF MULTIPLICATION IN DECIMALS.

For the purposes of these final tests, which were given in December 1938, I took five different forms or methods labelled A, B, C, D, and E below:

METHOD A. 42·613 × 21·3.

```
   42613
     213
 -------
 85226
   42613
  127839
 -------
 9076569

907·6569.  Ans.
```

METHOD B. 42·613 × 21·3.

```
  42 | 613
     |  213
 ----+-----
 852 | 26
  42 | 613
  12 | 7839
 ----+-----
 907 | 6569   Ans.
```

METHOD C. 42·613 × 21·3.

```
   42·613
      21·3
 ---------
 852·26
   42·613
   12·783 9
 ---------
 907·656 9   Ans.
```

METHOD D. 42·613 × 21·3.

```
  42·613
  21·3
 --------
 852·26
  42·613
  12·7839
 --------
 907·6569   Ans.
```

METHOD E. 42·613 × 21·3.

```
 426·13
   2·13
 --------
 852·26
  42·613
  12·7839
 --------
 907·6569   Ans.
```

It will be noticed that standard form (Method E) enters the "lists" for experiment for the first time in this investigation. For the purposes of the preliminary tests I felt that it might be a little too difficult for the comprehension of senior school children subjected to rapid testing. This was owing to the fact that the method of "Tracking the Unit" in addition to juggling with the decimal positions for standard form had to be taught before a test could be given. Now, however, I decided to include it in order to obtain as full a comparison as possible.

Five classes, with their teachers as instructors and invigilators, were chosen for the tests, and, in order to overcome the factor that multiplication improves with every successive test, the tests were given according to the following time-table and rota:

Class	V	IV	III	II	I
Monday - -	A	B	C	D	E
Tuesday - -	B	C	D	E	A
Wednesday -	C	D	E	A	B
Thursday - -	D	E	A	B	C
Friday - -	E	A	B	C	D

Certain of the pupils had had previous experience in the multiplication of decimals according to Method A. The extent of such experience is detailed below.

Class V. Nil.

Class IV. 22 out of 43 pupils had had some little experience in the contributory junior school from which they had been promoted. The extent of this experience, however, was so small, and so much time had elapsed (five months) since any call had been made upon it, that for the purposes of testing it could be ignored.

Class III. In this class of 44 pupils 15 were not acquainted with the multiplication of decimals. All the remainder were more or less acquainted with Method A.

Class II. 11 of this class of 41 pupils had no previous experience of the multiplication of decimals. The rest were fairly well acquainted with Method A.

Class I. All the 21 pupils of this class were reasonably well acquainted with Method A.

Some time before the tests were given I explained to the children and the teachers that they were the lambs being sacrificed on the altar of experiment but that the results of the "misery" to which they were being subjected would very probably make matters easier for their successors. I believe that as a result of this talk they showed more than average keenness when carrying out the tests.

It was arranged that the same amount of instruction with the same examples should be given to each class for each of the methods. Immediately after the instruction the tests were given and the time taken for each pupil to complete the test was noted. The instructions, and instruction and test examples are given on pp. 100-101.

Marking the Tests.

Each incorrect example was examined for the causes of error. There were six types of errors. These and the number of each of them occurring are given in the tabulated results of the tests (see pp. 98-99). It must be noted that not more than one multiplication error was counted for each partial product showing errors of multiplication, and not more than one addition error for each example showing errors of addition. In fact it would have been impossible to have done otherwise for I was unable to ascertain how far-reaching was the extent of each error.

It will be noticed that in counting errors of bad alignment no distinction was made between ordinary bad alignment and bad alignment due to the spatial arrangement of the decimal points. It was very difficult to make this distinction and in any case practically all the bad alignment appeared to be due either to the vertical line in Method B or to the spatial arrangement of the decimal points in Method C. In these two methods there were 103 cases of bad alignment whilst in the other three methods there were only 11 cases all told.

Observations on the Results of the Tests.

(i) Method A gave a high degree of accuracy. It was 21·6 per cent. more accurate than Method C, which gave the next best degree of accuracy, viz. 60·1 per cent. Even Class V, an 11 + C and D group of senior school children, obtained an over 50 per cent. result.

(ii) Method B—a modification of Method C intended as a means of overcoming the bad alignment of this form—did not give as good a result as Method C. As will be seen from the tabulated results, Method B gave the worse results in matters of degree of accuracy, alignment, and time factor.

(iii) Method D—Tracking the Unit Method—provided results which were least accurate. Before commencing these tests I felt that this honour would most definitely belong to standard form (Method E). Standard form, however, although a little slower, was 16·7 per cent. more accurate than the Tracking the Unit Method. The slowness of Methods E over D was, no doubt, due to the initial arrangement that, in Method E, was required with the decimal points. Its greater accuracy was obviously due to the greater ease of positioning the partial products. Thus the method of standard form should always be preferred to that of Tracking the Unit. This was the opinion of the invigilating teachers, who examined the papers with me, and myself.

(iv) Method C is the most untidy and Method E the most elegant of all five forms. If a method of multiplication of decimals, capable of being used for contracted work, is required, the question arises as to which of these two forms should be adopted. My own opinion, based largely on my knowledge of the pupils tested and my examination of the examples submitted in the tests, is that with further instruction and practice the rate of improvement with Method E would be very much higher than with Method C. In such circumstances, therefore, I would recommend the adoption of standard form.

(v) The number of mechanical errors occurring in the tests gives an approximate idea of the extent of the difficulties of each of the methods. In other words, the more difficult the method the greater will be the number of mechanical errors made by the pupil. I have totalled the number of errors of multiplication, addition, and omission. These totals will be found in the right-hand column of the tabulated results (see pp. 98-99). This approximate measure of

the efficiency of the methods shows that Method E presents little more difficulty than Method C.

(vi) A very brief reference summary of the results is given below :

Percentage correct.

V	IV	III	II	I	Total	
D	D	D	D	D	D	(36%)
E	E	E	E	E	E	(42·6%)
B	B	C} very	B	C	B	(56·7%)
C	C	B} close	C	A} very	C	(60·1%)
A	A	A	A	B} close	A	(73·2%)

Time taken.

E	D} the	C	E	E	E	($16\frac{1}{2}$ min.)
B	B} same	E	B	D	B	(16 „)
D	E	D	D	B	D	($15\frac{1}{2}$ „)
C	C	B	A	C	C	($14\frac{1}{2}$ „)
A	A	A	C	A	A	($12\frac{1}{2}$ „)

Results of Final Comparative Tests on Multiplication of Decimals.

Class	No. of Papers	No. of Examples	No. Correct	Average % Correct	Total Time (Min.)	Average Time per Paper (Min.)	D	Al	M	O	A	P	M+O+A	Method
V	29	145	77	53·3	557	19·2	23	2	56	3	7	—		Method A
IV	38	190	151	79·5	495	13·0	4	—	34	1	3	1		
III	39	195	146	74·9	421	10·8	9	1	42	—	10	—		
II	36	180	135	75·0	405	11·3	5	1	30	3	7	—		
I	19	95	80	84·2	133	7·0	—	—	13	—	4	—		
Total	161	805	589	73·2	2011	12·5	41	4	175	7	31	1	213	
V	29	145	44	30·3	710	24·5	46	6	69	10	9	1		Method B
IV	39	195	110	56·4	724	18·6	23	27	31	—	7	1		
III	38	190	102	53·7	500	13·2	37	11	57	—	6	5		
II	35	175	114	65·1	487	13·9	17	8	41	1	9	2		
I	21	105	89	84·8	175	8·3	3	1	10	—	4	—		
Total	162	810	459	56·7	2596	16·0	126	53	208	11	35	9	246	

V	29	145	57	39·3	597	20·6	22	16	102	6	5	3		Method C
IV	39	195	125	64·1	669	17·1	1	11	42	4	13	—		
III	40	200	106	53·0	575	14·4	46	12	51	1	14	1		
II	34	170	119	70·0	366	10·8	11	8	42	2	8	—		
I	21	105	83	79·0	153	7·3	5	3	14	—	2	1		
Total	163	815	490	60·1	2360	14·5	85	50	251	13	42	5	306	

V	29	145	31	21·4	645	22·3	90	3	77	7	9	5		Method D
IV	39	195	80	41·0	725	18·6	94	—	53	2	7	6		
III	39	195	62	31·8	534	13·7	91	1	72	3	14	4		
II	39	195	75	38·4	488	12·5	72	2	72	4	5	—		
I	20	100	51	51·0	184	9·2	28	—	23	3	6	4		
Total	166	830	299	36·0	2576	15·5	375	6	297	19	41	19	357	

V	29	145	40	27·6	723	24·9	85	—	76	1	6	—		Method E
IV	39	195	101	51·8	712	18·2	64	1	42	5	11	5		
III	40	200	85	43·6	555	13·9	84	—	45	11	7	2		
II	37	185	72	38·9	530	14·3	69	—	72	4	7	2		
I	19	95	51	53·7	189	9·9	26	—	35	1	1	—		
Total	164	820	349	42·6	2709	16·5	328	1	270	22	32	9	324	

D—errors due to incorrect positioning of the decimal point.

Al—errors due to bad alignment.

M—errors of multiplication.

O—errors of omission.

A—errors of addition.

P—errors due to incorrect positioning of the first figures of partial products subsequent to the first partial product.

Instruction lessons in preparation for the Final Comparative Tests.

Method A. Ignore the decimal points and proceed as in ordinary multiplication. To obtain the decimal point in the answer, count off from the right-hand figure of the product the same number of places as the total of the decimal places in the two factors.

N.B.—Do not put in the decimal places when carrying out the multiplication (see Method A on page 95).

Method B. Write down the multiplicand without the decimal point. Write down the multiplier (without its decimal point) so that its unit figure is beneath the last figure of the multiplicand. Draw a vertical line between the unit figure and the first decimal place position of the multiplicand. Extend it to four or five spaces beneath the multiplier. Position the partial products as in the multiplication of whole numbers.

METHOD C. Place the unit figure of the multiplier beneath the right-hand figure of the multiplicand. The decimal points in the partial products are always directly beneath that of the multiplicand. The partial products will be positioned as in the multiplication of whole numbers.

METHOD D. Place the decimal point of the multiplier beneath that of the multiplicand. The first figure of the first partial product is moved as many places to the left or right of the right-hand figure of the multiplicand as the multiplying figure is to the left or right of the unit figure of the multiplier. All other figures are positioned in accordance with the first partial product.

METHOD E. In this method the multiplier has to be changed so that its left-hand figure is a unit figure. Thus the decimal point has to be moved to the left or to the right. To compensate for this the decimal point of the multiplicand must be moved an equivalent number of places to the right or left, thus:

$$23{\cdot}\overset{\rightarrow}{6}12 \times 4\overset{\leftarrow}{2}{\cdot}1 \quad \text{becomes}$$
$$236{\cdot}12 \quad \times 4{\cdot}21.$$

Now as we are multiplying by the unit figure first, the first figure of the first partial product will always be beneath the right-hand figure of the multiplicand.

INSTRUCTION EXAMPLES.

The following examples were used throughout the whole series of instruction lessons.

N.B.—The pupils were instructed to place a nought to the left of the decimal point of any factor which did not possess whole numbers.

1. $23{\cdot}612 \times 42{\cdot}1$.
2. $42{\cdot}15 \times 3{\cdot}16$
3. $32{\cdot}614 \times {\cdot}213$.
4. $371{\cdot}24 \times {\cdot}042$.

NOTE ON FINAL COMPARATIVE TESTS' EXAMPLES (SEE BELOW).

The first examples in Monday's and Tuesday's tests were altered for the purposes of standard form to read, respectively, as follows:

A. 1. $62{\cdot}5 \times {\cdot}34$.	B. 1. $53{\cdot}4 \times {\cdot}26$.
A. 1. $53{\cdot}4 \times {\cdot}26$.	B. 1. $62{\cdot}5 \times {\cdot}34$.

Each pupil retained the same letters, A or B, throughout the week's tests.

Final Comparative Test Examples.

Monday.

A. 1.	6·25 × 3·4.	B. 1.	5·34 × 2·6.
2.	43·7 × 21·6.	2.	62·4 × 31·4.
3.	723·6 × 623·4.	3.	632·7 × 425·3.
4.	81·35 × ·417.	4.	64·35 × ·619.
5.	7·635 × ·036.	5.	9·245 × ·043.

Tuesday.

A. 1.	5·34 × 2·6.	B. 1.	6·25 × 3·4.
2.	62·4 × 31·4.	2.	43·7 × 21·6.
3.	632·7 × 425·3.	3.	723·6 × 623·4.
4.	64·35 × ·619.	4.	81·35 × ·417.
5.	9·245 × ·043.	5.	7·635 × ·036.

Wednesday.

A. 1.	62·5 × ·26.	B. 1.	53·4 × ·34.
2.	43·7 × 31·4.	2.	62·4 × 21·6.
3.	632·7 × 623·4.	3.	723·6 × 425·3.
4.	81·35 × ·619.	4.	64·35 × ·417.
5.	9·635 × ·043.	5.	7·245 × ·036.

Thursday.

A. 1.	53·4 × ·34.	B. 1.	62·5 × ·26.
2.	62·4 × 21·6.	2.	43·7 × 31·4.
3.	723·6 × 425·3.	3.	632·7 × 623·4.
4.	64·35 × ·417.	4.	81·35 × ·619.
5.	7·245 × ·036.	5.	9·635 × ·043.

Friday.

A. 1.	72·5 × ·34.	B. 1.	43·5 × ·26.
2.	93·6 × 21·7.	2.	85·3 × 31·4.
3.	862·7 × 523·4.	3.	726·3 × 432·5.
4.	93·15 × ·618.	4.	74·35 × ·617.
5.	9·246 × ·056.	5.	7·638 × ·047.

N.B.—None of the test examples consisted of specially extreme cases, *e.g.* ·0041 × ·00063. A reasonably wide range was adopted with multiplicands and multipliers. It will be noticed that, in both cases, the examples range from 1 to 3 places either side of the decimal point.

The popularity of the different methods of Multiplication of Decimals in various types of schools.

Before carrying out my comparative tests on the multiplication of decimals I made enquiries in a number of schools in the Black Country area concerning the methods they used for the multiplication of decimals. All of them used one of the four methods A, C, D, or E as set out on p. 95.

Below is (*a*) a list giving the aggregate number of schools using the different methods, and (*b*) a list in which the schools are sectionalised according to type.

(*a*)		Number of schools using	Method	A	19
		,, ,, ,, ,,	,,	C	4
		,, ,, ,, ,,	,,	D	2
		,, ,, ,, ,,	,,	E	3
(*b*)	i.	Senior schools using	Method	A	10
		,, ,, ,, ,,	,,	D	1
		,, ,, ,, ,,	,,	E	1
	ii.	Central schools using	Method	A	4
		,, ,, ,, ,,	,,	C	1
		,, ,, ,, ,,	,,	D	1
		,, ,, ,, ,,	,,	E	1
	iii.	Technical schools using	Method	C	2
		,, ,, ,, ,,	,,	E	1
	iv.	Secondary schools using	Method	A	5
		,, ,, ,, ,,	,,	C	1

The enquiry was by no means exhaustive but it served as a basis on which I was able to build up my investigation. The replies to my enquiry showed that until recently one central and two secondary schools had used Method E, and one central school had used Method C. All of these schools have now adopted Method A.

In one girls' central school there was a certain amount of controversy about the relative merits of Methods C and D between two lady teachers both of whom were partially responsible for the teaching of mathematics. The Principal of the school—a specialist in geography—found it difficult to give a definite ruling. As a result each lady endeavoured to bring about the adoption of her own method. In such instances, and there must be several, the results of this investigation should be of value. H. W.

GLEANINGS FAR AND NEAR

1310. Under the reasonings of a new geometry and the now acknowledged curvature of our standard of straightness, concurrent rays may cease to be parallel and finally may meet; but the application of this scientific truth to problems of international politics has not yet been made.—*Times*, January 17, 1939, p. 12. [Per Mr. A. J. Ede; Dr. J. Wishart.]

ON CHORDS OF A CONIC WHICH TOUCH ANOTHER CONIC.

By R. L. Goodstein.

A closed conic C is completely contained inside a closed conic S. From any point 1 on S we draw a tangent 1, 2 to C meeting S again at 2. From 2 we draw the tangent 2, 3 meeting S again at 3, and so on; for any n, the tangent to C from the point n on S meets S again at $n+1$.

A pair of points p, q determines two arcs of S. We shall show that if the points 1, 2, 3, ... n do not form a closed polygon for any n, then there are points of the sequence 1, 2, 3, ... on both the arcs determined by any pair p, q.*

We shall say that the pair of points (p, q) *separates* the pair (m, n) if m and n lie one in each of the arcs determined by p, q. If (p, q) separates (m, n) we write

$$(p, q \between m, n).$$

We determine a *direction* round C by attaching to C, at the point of contact of the tangent 1, 2, an arrow in the direction from the point 1 to the point 2; the direction along C so determined we call the positive direction along C. We observe that if Q is the point of contact of C with the tangent n, $n+1$, then the positive direction round C, at the point Q, is from n to $n+1$; for C is contained between the tangents 1, 2 and 2, 3, and so, since the positive direction is from 1 to 2, the direction 2 to 3 is also positive for C, and furthermore, for any n, C is contained between the tangents $n-1$, n and n, $n+1$, and so, if the direction $n-1$ to n is positive so too is the direction n to $n+1$.

If the point of contact of a variable tangent t to C describes C in the positive direction the rotation of t is said to be in the positive direction.

A variable tangent TT' rotates in the positive direction from an initial position p, $p+1$ to a final position q, $q+1$. The arc of S described by T is called the arc p. q; similarly, if TT' rotates in the positive direction from q, $q+1$ to p, $p+1$, the arc of S described by T is called the arc q, p. It follows that as T describes the arc p, q then T' describes the arc $p+1$, $q+1$, and as T describes the arc q, p, T' describes the arc $q+1$, $p+1$. Therefore if $(m, n \between p, q)$, where m lies on the arc p, q and n on the arc q, p, then $m+1$ lies on the arc $p+1$, $q+1$ and $n+1$ on the arc $q+1$, $p+1$ so that

$$(m+1, n+1 \between p+1, q+1);$$

whence we see that $(m, n \between p, q)$ implies $(m+r, n+r \between p+r, q+r)$ for any r. In particular, if for each n we can find p_n, q_n so that $(1, n \between p_n, q_n)$, then given any r, s we can find p', q' so that

* Mr. H. D. Ursell has proved the stronger result that if the polygon 1, 2, 3, ... n is not closed then the set of points is dense-in-itself.

$(r, s \between p', q')$; for if $r < s$, and if we denote p_{s-r+1} by p and q_{s-r+1} by q, we have $(1, s-r+1 \between p, q)$ and therefore $(r, s \between p+r-1, q+r-1)$. Thus to prove our theorem it suffices to show that given any n we can find p_n, q_n such that $(1, n \between p_n, q_n)$.

We show, next, that if the points a, $a+b$, $a+2b$, $a+3b$, ... lie in the same direction along an arc p, q of S, then the sequence is necessarily finite. Let $f_n(s)$ denote the length of the arc 1, $n+1$, where s is the length of the arc of S from a fixed point 0 to the point 1. It is readily seen that for a fixed n, $f_n(s)$ is a continuous function of s. Let L_n be the lower bound of $f_n(s)$, so that $L_n \geqslant 0$; since $f_n(s)$ is continuous we can find s' such that $f_n(s') = L_n$. Thus if the polygon 1, 2, 3, ... n is not closed, $f_n(s') \neq 0$ and so $L_n > 0$.

Let α_r be the length of the arc 0, $a+rb$, then $f_b(\alpha_n)$ is the length of the arc $a+rb$, $a+(r+1)b$, and so the sum of the lengths of the arcs a, $a+b$; $a+b$, $a+2b$; ...; $a+(n-1)b$, $a+nb$ is greater than nL_b. Hence, since L_b is not zero, we can find n, so that nL_b is greater than the length of the arc p, q, and so the sequence a, $a+b$, $a+2b$, ... must terminate before the point $a+nb$.

Consider the sequence of points, 1, $n+1$, $2n+1$, $3n+1$, ... The four points 1, $n+1$, $2n+1$, $3n+1$ may be arranged in three ways:

$$\text{(i)} \quad (1, n+1 \between 2n+1, 3n+1);$$

$$\text{(ii)} \quad (1, 2n+1 \between n+1, 3n+1);$$

$$\text{(iii)} \quad (1, 3n+1 \between n+1, 2n+1).$$

For those values of n for which (i) holds our theorem is established. Case (iii) cannot in fact occur, for if

$$(1, 3n+1 \between n+1, 2n+1),$$

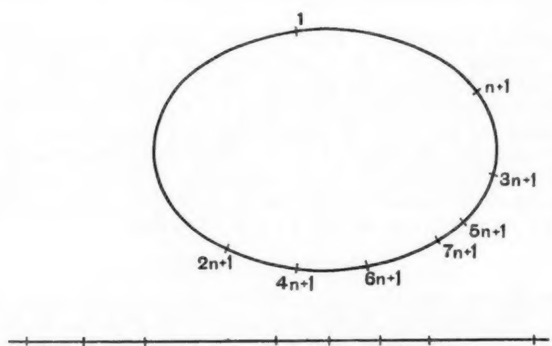


Fig. 1.

then $\qquad (n+1,\ 4n+1 \between 2n+1,\ 3n+1)$

and $\qquad (2n+1,\ 5n+1 \between 3n+1,\ 4n+1),$

$\qquad (3n+1,\ 6n+1 \between 4n+1,\ 5n+1),$

and so on indefinitely. Thus we obtain an *infinite* sequence of points $2n+1$, $4n+1$, $6n+1$, ... , each following its predecessor in the same direction along S and all contained in the arc $2n+1$, $n+1$, and this we have seen to be impossible.

It remains to consider case (ii).

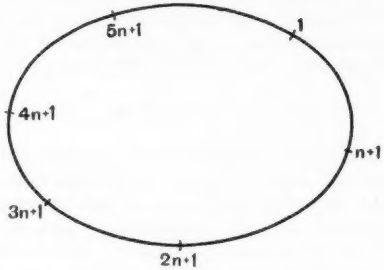


FIG. 2.

If $\qquad (1,\ 2n+1 \between n+1,\ 3n+1),$

then $\qquad (n+1,\ 3n+1 \between 2n+1,\ 4n+1),$

so that either $\qquad (1,\ n+1 \between 2n+1,\ 4n+1),$

in which case the theorem is proved,

or $\qquad (1,\ 3n+1 \between 2n+1,\ 4n+1),$

whence $\qquad (n+1,\ 4n+1 \between 3n+1,\ 5n+1)$

and so again,

either $\qquad (1,\ n+1 \between 3n+1,\ 5n+1),$ proving the theorem,

or $\qquad (1,\ 4n+1 \between 3n+1,\ 5n+1).$

Proceeding in this way, we either find a pair p, q such that $(1, n+1 \between p, q)$ or we determine a sequence of points $n+1$, $2n+1$, $3n+1$, ... , each following its predecessor in the same direction along S and all contained in the arc $n+1$, 1. We have seen that this sequence must terminate at $rn+1$, say, so that

$$(1,\ n+1 \between \overline{s-1}\,n,\ \overline{s+1}\,n), \quad s \leqslant r,$$

and the theorem is proved. R. L. G.

1311. "WINNING WAYS."

A comedy of Welsh small-town life. In which the Eternal Triangle becomes increasingly Obtuse, and owing to the addition of an Unknown Quantity to the Problem, ends its career as a Straight line.—*Radio Times*—Programme for Friday, April 7. [Per Mr. F. J. Wood; Dr. G. R. Langdale.]

MATHEMATICAL NOTES

1443. *Note on certain identities.*

Having read Dr. Lidstone's Note (No. 1418), kindly sent me by the author, it occurred to me that the identities under discussion may be obtained by various simple transformations of the integral

$$I = r\binom{n}{r}\int_0^p x^{r-1}(1-x)^{n-r}\,dx, \quad \text{......................(1)}$$

where, in order to remain on the ground covered by the discussion, it may be assumed that n and r are positive integers, $n \geqslant r$, and that $0<p<1$. Various simple transformations of (1), taking into account the binomial theorem and the Beta-integral, viz.

$$r\binom{n}{r}\int_0^1 x^{r-1}(1-x)^{n-r}\,dx = 1, \quad \text{......................(2)}$$

lead to expansions which, compared two and two, furnish as many algebraic identities. Some of these have already been considered by others; besides the contributors to the *Mathematical Gazette* (which is not accessible here in Copenhagen) I may mention Laplace, Karl Pearson, H. E. Soper and R. Frisch *; but perhaps the remark that (1) may be regarded as the common source of all these identities is new.

I shall begin by stating six expansions of (1) and give the proofs afterwards. Let q be defined by

$$p+q=1\,; \quad \text{..............................(3)}$$

then these expansions are

$$I = r\binom{n}{r}\sum_{s=0}^{n-r}(-1)^s\binom{n-r}{s}\frac{p^{r+s}}{r+s} \quad \text{..........................(4)}$$

$$= 1 - r\binom{n}{r}\sum_{s=0}^{r-1}(-1)^s\binom{r-1}{s}\frac{q^{n-r+s+1}}{n-r+s+1} \quad \text{.............(5)}$$

$$= p^r\sum_{s=0}^{n-r}\binom{r+s-1}{s}q^s \quad \text{.......................................(6)}$$

$$= 1 - q^{n-r+1}\sum_{s=0}^{r-1}\binom{n-r+s}{s}p^s \quad \text{..........................(7)}$$

$$= \sum_{s=r}^{n}\binom{n}{s}p^s q^{n-s} \quad \text{..(8)}$$

$$= 1 - \sum_{s=0}^{r-1}\binom{n}{s}p^s q^{n-s}. \quad \text{.......................................(9)}$$

Of these, (4) is obtained simply by expanding the binomial $(1-x)^{n-r}$ in (1) and integrating term by term.

* See the literature quoted in Ragnar Frisch: *Sur les semi-invariants et moments employés dans l'étude des distributions statistiques* (Oslo, 1926), p. 49.

In order to obtain (5) we observe that $\int_0^p = \int_0^1 - \int_p^1$, whence (1) becomes, by (2),

$$I = 1 - r\binom{n}{r}\int_p^1 x^{r-1}(1-x)^{n-r}\,dx,$$

or, putting $x = 1-y$,

$$I = 1 - r\binom{n}{r}\int_0^q (1-y)^{r-1}y^{n-r}\,dy. \quad \text{.................(10)}$$

Developing now the binomial $(1-y)^{r-1}$ and integrating term by term, we obtain (5).

If, in (1), we put $x = tp$, we find

$$I = r\binom{n}{r}p^r\int_0^1 t^{r-1}(\overline{1-t}+tq)^{n-r}\,dt,$$

whence (6) is obtained by expanding the binomial, evaluating the Beta-integrals and reducing.

(7) is proved by putting $y = tq$ in (10), whence

$$I = 1 - r\binom{n}{r}q^{n-r+1}\int_0^1 (\overline{1-t}+tp)^{r-1}t^{n-r}\,dt.$$

Developing the binomial and evaluating the Beta-integrals, we have (7).

To obtain (8), we put, in (1), $x = p(1-t)$, and find

$$I = r\binom{n}{r}p^r\int_0^1 (1-t)^{r-1}(q+pt)^{n-r}\,dt;$$

whence, developing $(q+pt)^{n-r}$ and evaluating the Beta-integrals, (8) results.

The expansion (9) follows from (8) by expanding $(p+q)^n$.

Comparing the six expansions (4)-(9) two and two, we obtain $\binom{6}{2}$ or 15 algebraic identities, of which, however, the one obtained from (8) and (9) is trivial, being the binomial theorem itself. In particular, we obtain from (6) and (7) for $n = 2r-1$

$$1 = \sum_{s=0}^{r-1}\binom{r+s-1}{s}(p^r q^s + q^r p^s),$$

or, putting $p = \alpha/(\alpha+\beta)$, $q = \beta/(\alpha+\beta)$,

$$(\alpha+\beta)^r = \sum_{s=0}^{r-1}\binom{r+s-1}{s}(\alpha^{r-s}+\beta^{r-s})\left(\frac{\alpha\beta}{\alpha+\beta}\right)^s,$$

which is the identity from which the discussion has arisen.

J. F. STEFFENSEN.

1444. *Practice.*

When we have to find the value of a number of articles at a given price or multiply a compound quantity by a whole number, we may

employ the method of " Simple Practice ". In reality we are changing the question to another one with the same answer. Thus if we find the cost of 365 days' wages at £1 14s. 7d. per day, instead of multiplying £1 14s. 7d. by 365, we change the question to one of multiplying £365 by $(1+\frac{14}{20}+\frac{7}{240})$, using aliquot parts.

In " Compound Practice " the commutation is not usually made in modern work, but a simpler method than that usually taught is available. It is best illustrated by an example.

To find the cost of 3 tons 13 cwts. 3 qrs. at £4 7s. 3d. per ton.

	£	s.	d.		£	s.	d.
1 ton costs	4	7	3	3 tons cost	13	1	9
10 cwt. ,,	2	3	7·5	10 cwt. ,,	2	3	7·5
1 cwt. ,,		4	4·35	3 cwt. ,,		13	1·05
1 qr. ,,		1	1·09	3 qrs. ,,		3	3·27
					£16	1	8·82
					=£16	1	9 nearly.

The method consists of finding the cost of a unit of each denomination and multiplying by the number of units. It is an application of the unitary method and could be used in conjunction with a ready-reckoner, or the compound quantities may be replaced by decimals.

The method is given, so far as the author is aware, only in Hughes Dowling's *Instructive Arithmetic*, Book III, under the title, " Method of Denominational Units ". T. W. HALL.

1445. *Feuerbach's Theorem.*

AD is an altitude of the triangle ABC, circumcentre O, incentre I; $\alpha I\alpha'$ is equal and parallel to AD, N is the mid-point of AO. Draw AU perpendicular to OI, IL perpendicular to OA and let NU cut $\alpha I\alpha'$ at J.

Then since AI bisects the angle OAD and A, α, L, I, U lie on a circle,

$$\angle \alpha UA = \angle \alpha IA = \angle IAD = \angle IAN.$$

Thus
$$\begin{aligned}\angle NU\alpha &= \angle NUA - \angle \alpha UA \\ &= \angle NAU - \angle NAI \\ &= \angle IAU \\ &= \angle I\alpha U.\end{aligned}$$

Hence
$$JU = J\alpha,$$

so the circle on AO as diameter touches the circle with centre J and radius $J\alpha$ at U.

Again, if AK is perpendicular to the common tangent at U,

$$A\alpha^2 = 2AK \,.\, NJ\,;$$

$$AU^2 = AK \,.\, AO \text{ by similar triangles};$$

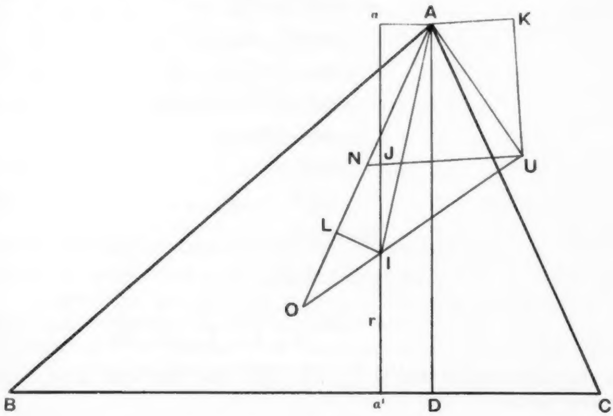
and $\qquad IL = A\alpha$ (chords subtending equal angles at the circumference).

Thus $OI^2/OA^2 = IL^2/AU^2 = A\alpha^2/AU^2 = 2NJ/OA$;

but $\qquad OI^2 = R^2 - 2Rr.$

Therefore $NJ = \frac{1}{2}R - r.$

Thus $\qquad JU = r.$



Hence the circle, centre J, radius $J\alpha$, is the reflection of the incircle in the line joining the mid-points of AB, AC ; but the circle on OA as diameter is the reflection of the nine-points circle in the same line.

Therefore the incircle touches the nine-points circle at a point V which is the reflection of U in the line joining the mid-points of AB, AC.

Those familiar with the properties of pedal lines may easily see that V is the orthopole of the diameter OI ; that is, the pedal lines of the extremities of this diameter meet at V. E. P. Lewis.

1446. *Arithmetical flukes.*

1. From a figure, or by calculation,

$$\cot 7\tfrac{1}{2}^\circ = \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}$$
$$> 7{\cdot}59575$$
$$> 7{\cdot}59.$$

2. From a figure

$$\tfrac{1}{2}n \sin(2\pi/n) < \pi < n \tan(\pi/n).$$

3. Putting $n=24$ in the right-hand side of this inequality, we have

$$\pi<24 \tan 7\tfrac{1}{2}°<24/7{\cdot}59=8/2{\cdot}53.$$

Hence $$\pi^2<64/6{\cdot}4009<10<(3\tfrac{1}{6})^2. \qquad \text{(i)}$$

Putting $n=48$ in the left-hand side of the inequality, we have

$$\pi>24 \sin 7\tfrac{1}{2}°.$$

Thus
$$\begin{aligned}\pi^2&>576 \sin^2 7\tfrac{1}{2}°\\ &=288(1-\cos 15°)\\ &=288-72(\sqrt{6}+\sqrt{2})\\ &>288-(72\times 3{\cdot}86375)\\ &=288-278{\cdot}19\\ &=9{\cdot}81\\ &>(3\tfrac{1}{8})^2 \qquad \text{(ii)}\end{aligned}$$

Hence $$3\tfrac{1}{8}<\pi<3\tfrac{1}{6}.$$

4. $$\begin{aligned}\cot 7\tfrac{1}{2}°&>7{\cdot}59575\\ &=7+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{9+}\,\frac{1}{85}, \qquad \text{(iii)}\end{aligned}$$

and the convergents are 8, $7\tfrac{1}{2}$, $7\tfrac{3}{5}$, $7\tfrac{28}{47}$; the last one is too small but very near.

Hence $$\cot 7\tfrac{1}{2}°>357/47.$$

Hence $$\operatorname{cosec}^2 7\tfrac{1}{2}°>1+(357/47)^2=129658/47^2.$$

Hence $$\operatorname{cosec} 7\tfrac{1}{2}°>360{\cdot}08/47, \qquad \text{(iv)}$$

and $$\begin{aligned}\cot 3\tfrac{3}{4}°&\equiv\cot 7\tfrac{1}{2}°+\operatorname{cosec} 7\tfrac{1}{2}°\\ &>717{\cdot}08/47\\ &>15{\cdot}257. \qquad \text{(v)}\end{aligned}$$

Also $$\begin{aligned}\operatorname{cosec}^2 3\tfrac{3}{4}°&>1+(15{\cdot}257)^2\\ &=233{\cdot}776049.\end{aligned}$$

Thus $$\operatorname{cosec} 3\tfrac{3}{4}°>15{\cdot}289,$$

and so $$\begin{aligned}\cot 1\tfrac{7}{8}°&>30{\cdot}546\\ &>30\tfrac{6}{11}. \qquad \text{(vi)}\end{aligned}$$

That is, $$\begin{aligned}\pi&<\frac{96}{30\frac{6}{11}}\\ &=\frac{96\times 11}{336}\\ &=3\tfrac{1}{7}.\end{aligned}$$

5. We have also

$$\cos^2 7\tfrac{1}{2}^\circ = \tfrac{1}{2}(1+\cos 15^\circ)$$
$$= \tfrac{1}{2} + \tfrac{1}{8}(\sqrt{6}+\sqrt{2})$$
$$< \cdot 982983. \quad \text{.............................(vii)}$$

Thus $\cos 7\tfrac{1}{2}^\circ < \cdot 991445.$ ....(viii)

Hence $\cos^2 3\tfrac{3}{4}^\circ < \cdot 9957225$

and $\cos 3\tfrac{3}{4}^\circ < \cdot 997859,$ ....(ix)

whence $1 - \cos 3\tfrac{3}{4}^\circ > \cdot 002141.$

Now $\pi > 96 \sin 1\tfrac{7}{8}^\circ;$

hence $\pi^2 > 96 \times 48 \times \cdot 002141$

$= 9\cdot 865728.$

Thus $\pi > 3\cdot 1409 > 3\tfrac{10}{71}.$

Hence $3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}.$ N. M. Gibbins.

1447. *Triangle bordé de triangles isoscèles semblables.*

Dans un intéressant article, le Professeur Gino Loria a retrouvé cette proposition connue : *les centres des triangles équilatéraux construits extérieurement ou intérieurement sur les côtés d'un triangle quelconque sont les sommets d'un nouveau triangle équilatéral.* Puis, il s'est posé la question de savoir si des théorèmes analogues ont lieu en construisant sur les côtés du triangle donné des polygones réguliers d'un nombre de côtés plus grand que trois.*

Plus généralement, nous allons établir la condition nécessaire et suffisante pour que le triangle formé par les sommets de triangles isoscèles semblables, construits extérieurement, ou intérieurement, sur les côtés du triangle fondamental, soit équilatéral.

Soient X, Y, Z les sommets des triangles isoscèles BXC, CYA, AZB construits extérieurement, ou intérieurement, sur les côtés $BC=a$, $CA=b$, $AB=c$ d'un triangle ABC, 2α l'angle au sommet commun à ces trois triangles. Pour que le triangle XYZ soit équilatéral, il faut et il suffit que $YZ=ZX=XY$, c'est-à-dire que l'on ait

$$YZ^2 = AY^2 + AZ^2 - 2AY\,.\,AZ \cos YAZ$$
$$= ZX^2 = BZ^2 + BX^2 - 2BZ\,.\,BX \cos ZAX$$
$$= XY^2 = CX^2 + CY^2 - 2CX\,.\,CY \cos CYA.$$

Ces relations se réduisent aisément aux suivantes :

$$a \cos B\,(2\cos 2\alpha + 1) = b \cos A\,(2\cos 2\alpha + 1),$$
$$b \cos C\,(2\cos 2\alpha + 1) = c \cos B\,(2\cos 2\alpha + 1),$$
$$c \cos A\,(2\cos 2\alpha + 1) = a \cos C\,(2\cos 2\alpha + 1),$$

* *Mathematical Gazette*, XXIII, p. 366.

lesquelles exigent, à la fois, que

$$a \cos B = b \cos A,\ b \cos C = c \cos B,\ c \cos A = a \cos C, \quad \text{......(i)}$$

à moins que l'on ait

$$2 \cos 2\alpha + 1 = 0, \text{ soit } \alpha = 60°.$$

Les égalités (i), desquelles il résulte que

$$\cos A : \cos B = a : b = \sin A : \sin B, \ldots,$$

conduisent à l'hypothèse banale où $a = b = c$, c'est-à-dire que le triangle ABC est équilatéral.

Dès lors, *dans un triangle* ABC, dont les côtés ne sont pas tous égaux entre eux, *la condition nécessaire et suffisante pour que le triangle envisagé* XYZ *soit équilatéral, est que l'angle au sommet des triangles isoscèles* BXC, CYA, AZB *soit égal à* 120° ; autrement dit, *il faut et il suffit que les points* X, Y, Z *soient les centres des triangles équilatéraux construits extérieurement, ou intérieurement, sur les côtés du triangle* ABC. En particulier, les centres des polygones réguliers de n côtés, construits extérieurement, ou intérieurement, sur les côtés du triangle ABC, ne sont les sommets d'un triangle équilatéral que dans le cas où $n = 3$.

Le Mans, France. V. Thébault.

1448. *The circle in homogeneous (areal) coordinates.*

The following methods of dealing with the circle

$$S \equiv (lx + my + nz)(\Sigma x) - \Sigma a^2 yz = 0$$

appear simpler than those indicated in textbooks.

1°. *To interpret l, m, n geometrically.*

Suppose the circle cuts the side BC of the triangle of reference at $D_1(0, y_1, z_1)$ and $D_2(0, y_2, z_2)$. Then y_1/z_1 and y_2/z_2 are the two values of y/z given by

$$(my + nz)(y + z) - a^2 yz = 0.$$

Hence $$y_1/z_1 + y_2/z_2 = (a^2 - m - n)/m,$$

and $$y_1 y_2/z_1 z_2 = n/m.$$

Also $$y_1/z_1 = D_1C/BD_1, \quad y_2/z_2 = D_2C/BD_2.$$

Therefore $$1 + y_1/z_1 = a/BD_1, \quad 1 + y_2/z_2 = a/BD_2.$$

Hence $$\begin{aligned} a^2/(BD_1 \,.\, BD_2) &= (1 + y_1/z_1)(1 + y_2/z_2) \\ &= 1 + (a^2 - m - n)/m + n/m \\ &= a^2/m. \end{aligned}$$

Hence $m = BD_1 \,.\, BD_2 =$ *the power of B with respect to the circle.*

Similarly l and n are the powers of A and C respectively.

2°. *The power of the point* $P(x, y, z)$, where $x + y + z = 1$, with respect to the circle $S = 0$.

Any point R on PA is $(x + \lambda, y, z)$, where $\lambda = PR/RA$. Hence if PA cuts the circle at R_1 and R_2, then PR_1/R_1A and PR_2/R_2A are the

two values of λ given by the equation $S(x+\lambda, y, z)=0$.

Hence $$(PR_1/R_1A)(PR_2/R_2A)=S/l.$$

But $$AR_1 \,.\, AR_2 = l.$$

Therefore $$PR_1 \,.\, PR_2 = S.$$

Hence *the power of P is S.*

(Cf. Jones : *Introduction to Anal. Geom.*, page 480.)

3°. *The centre of the circle $S=0$.*

Let (x_1, y_1, z_1) be the *absolute* coordinates of the centre O of the circle ABC. Let $P(x_2, y_2, z_2)$ be the point at infinity in the direction perpendicular to the line $lx+my+nz=0$, so that

$$x_2=\tfrac{1}{2}\frac{\partial\Omega}{\partial l}, \qquad y_2=\tfrac{1}{2}\frac{\partial\Omega}{\partial m}, \qquad z_2=\tfrac{1}{2}\frac{\partial\Omega}{\partial n},$$

where $$\Omega \equiv \Sigma a^2l^2 - 2\Sigma\, bcmn \cos A.$$

Then, if W be the centre of $S=0$, W lies on OP, and therefore its coordinates are

$$(x_1+kx_2,\ y_1+ky_2,\ z_1+kz_2),$$

where k is such that W and P are conjugate points with respect to $S=0$.

The condition for conjugate points gives

$$lx_2+my_2+nz_2-\Sigma a^2[(y_1+ky_2)z_2+(z_1+kz_2)y_2]=0.$$

But since O and P are conjugate points with respect to the circle $\Sigma a^2yz=0$, we have

$$\Sigma a^2(y_1z_2+y_2z_1)=0.$$

Hence k is given by

$$lx_2+my_2+nz_2-2k\Sigma a^2y_2z_2=0.$$

But $$lx_2+my_2+nz_2=\Omega,$$

and $$\Sigma a^2y_2z_2=-4M^2\Omega,$$

where M is the area of the triangle ABC.

[For, $\Sigma a^2y_2z_2=0$ is the envelope equation of the circular points at infinity and hence $\Sigma a^2y_2z_2\equiv\mu\Omega$, where μ is independent of l, m, n.]

Hence $$k=-1/8M^2.$$

4°. *The radius ρ of the circle $S=0$.*

The power of W with respect to the circle $S=0$ is $(-\rho^2)$.

Hence

$$-\rho^2=lx_1+my_1+nz_1+k\Omega-\Sigma a^2(y_1+ky_2)(z_1+kz_2).$$

But $[-\Sigma a^2y_1z_1]=$ the power of O with respect to the circle $ABC=(-R^2)$,

$$\Sigma a^2(y_1z_2+y_2z_1)=0,$$

$$\Sigma a^2y_2z_2=-4M^2\Omega,$$

and $$k=-1/8M^2.$$

Hence $$\rho^2=R^2-(R/2M)(\Sigma al\cos A)+\Omega/16M^2.$$

(Cf. Salmon's *Conic Sections*, p. 129.)

F. H. V. GULASEKHARAM.

1449. *A new form of the equation of the director circle of a conic.*

If $\phi \equiv ul^2 + vm^2 + wn^2 + 2fmn + 2gnl + 2hlm = 0$ is the *areal* envelope equation of a conic, the equation of its director circle is

$$uD_{AA} + vD_{BB} + wD_{CC} + 2fD_{BC} + 2gD_{CA} + 2hD_{AB} = 0, \quad \text{......(1)}$$

where D_{AA} denotes the power of any point P with respect to the point circle at A; D_{BC} denotes the power of P with respect to the circle on BC as diameter; and so on. (ABC is the triangle of reference.)

I hit upon this useful form some years ago and have not seen it anywhere in print. On the general principle that an elegant result should be associated with an elegant proof, I made up the following proof:

Let p, q, r be the perpendicular distances of a straight line from A, B, C respectively. Then it is well known that

(i) $\Omega \equiv \Sigma a^2p^2 - 2\Sigma bcqr \cos A = 4M^2$, where M is the area of the triangle ABC;

(ii) the areal equation of the straight line is $px + qy + rz = 0$.

Let us call this straight line the line (p, q, r). Let two perpendicular tangents (p_1, q_1, r_1), (p_2, q_2, r_2) to the conic $\phi = 0$ intersect at $P(x, y, z)$. Call them PT_1, PT_2 respectively. Then with reference to PT_1, PT_2 as axes of *Cartesian* coordinates, A, B, C are the points (p_2, p_1), (q_2, q_1), (r_2, r_1) respectively.

Now since PT_1, PT_2 are tangents to $\phi = 0$, therefore

$$\phi(p_1, q_1, r_1) + \phi(p_2, q_2, r_2) = 0,$$

or

$$\Sigma u(p_1{}^2 + p_2{}^2) + 2\Sigma f(q_1r_2 + q_2r_1) = 0. \quad \text{................(2)}$$

But $$p_1{}^2 + p_2{}^2 = PA^2 = D_{AA};$$

and $q_1r_2 + q_2r_1 =$ the power of the origin (*i.e.* of P) with respect to the circle on BC as diameter $= D_{BC}$.

Hence the coordinates of the point P satisfy equation (1), which is therefore the equation of the director circle.

Notes:

(i) The *joint-director* circle of the two *confocal conics* $\phi = 0$ and $\phi - k\Omega(l, m, n) = 0$ is

$$\Sigma uD_{AA} + 2\Sigma fD_{BC} - 4kM^2(x + y + z)^2 = 0. \quad \text{.............(3)}$$

[For, in this case equation (2) above is replaced by

$$\phi(p_1, q_1, r_1) + \phi(p_2, q_2, r_2) - k\Omega(p_2, q_2, r_2) = 0.]$$

(ii) For the application of the equations (1) and (3), the following fundamental result is essential:

The power of $P(x, y, z)$, where $x + y + z = 1$, with respect to the circle

$$S \equiv (lx + my + nz)(\Sigma x) - \Sigma a^2yz = 0$$

is S. (*See* Note 1448.)

This gives $$D_{AA} = (c^2y + b^2z)(\Sigma x) - \Sigma a^2yz,$$
and $$D_{BC} = (bcx \cos A)(\Sigma x) - \Sigma a^2yz.$$

(iii) From (ii) above, the equation (1) of the director circle can be thrown into the form given in textbooks.

When $\phi = 0$ is a parabola,

$$u + v + w + 2f + 2g + 2h = 0,$$

and equation (1) reduces to

$$(l'x + m'y + n'z)(x + y + z) = 0,$$

where
$$l' = vc^2 + wb^2 + 2fbc \cos A,$$
$$m' = wa^2 + uc^2 + 2gca \cos B,$$
$$n' = ub^2 + va^2 + 2hab \cos C.$$

Thus when $\phi = 0$ is a parabola, its *directrix* is

$$l'x + m'y + n'z = 0,$$

where l', m', n' are proportional to the coefficients of x^2, y^2, z^2 in the expression on the left-hand side of equation (1).

(iv) I have also been able to deduce the form of the equation (1) by writing the equation of the circular lines through P (*i.e.* of the *point circle at* P) in the form

$$(PA^2 \,.\, x + PB^2 \,.\, y + PC^2 \,.\, z)(\Sigma x) - \Sigma a^2yz = 0,$$

which reduces to

$$\Sigma PA^2 \,.\, x^2 + 2\Sigma(PB^2 + PC^2 - a^2)yz = 0,$$

or $$\Sigma D_{AA}x^2 + 2\Sigma D_{BC}yz = 0,$$

and then expressing the condition that they should be conjugate lines with respect to the conic $\phi = 0$.

Examples:

(i) The director circle of the conic $ul^2 + vm^2 + wn^2 = 0$ is

$$u\, D_{AA} + v\, D_{BB} + w\, D_{CC} = 0,$$

which for all values of u, v, w cuts orthogonally the common radical circle of the circles $D_{AA} = 0$, $D_{BB} = 0$, $D_{CC} = 0$, viz. the circle ABC. Hence we have *Gaskin's Theorem*:

The circumcircle of a triangle self-polar with respect to a conic cuts orthogonally the director circle of the conic.

(2) The director circle of the in-conic $fmn + gnl + hlm = 0$ is

$$fD_{BC} + g\, D_{CA} + h\, D_{AB} = 0,$$

which for all values of f, g, h cuts orthogonally the common radical circle of the circles $D_{BC} = 0$, $D_{CA} = 0$, $D_{AB} = 0$, viz. the polar circle of the triangle ABC. Hence we have a new proof of the well-known theorem:

The polar circle of a triangle cuts orthogonally the director circles of all in-conics of the triangle.

(3) The envelope equation of the *Steiner ellipse* of the triangle ABC is

$$mn + nl + lm = 0.$$

Its director circle is

$$D_{BC} + D_{CA} + D_{AB} = 0,$$

or $$(\Sigma bcx \cos A)(\Sigma x) - 3\Sigma a^2yz = 0,$$

or $$\tfrac{1}{3}(\Sigma bcx \cos A)(\Sigma x) - \Sigma a^2yz = 0,$$

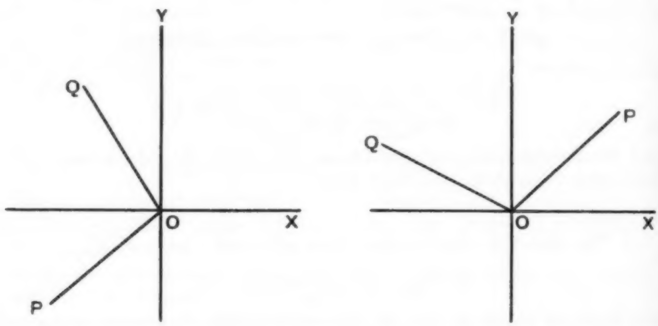
which proves that *the director circle of the Steiner ellipse, the circumcircle, the nine-points circle and the polar circle of the triangle ABC are coaxal.*

F. H. V. GULASEKHARAM.

1450. *The addition theorems.*

The following proof of the addition theorems is general and is easier to explain and easier for students to appreciate than the usual proof involving vectors. No reference to the method has been found in any modern textbook on elementary trigonometry.

It is most natural to begin by establishing the formula for $\sin(A - B)$.



Let A and B be angles of any magnitude and let OP and OQ be unit vectors such that $\angle XOP$ and $\angle XOQ$ are coterminal with A and B respectively. Then the point P is $(\cos A, \sin A)$, the point Q is $(\cos B, \sin B)$, $\angle QOP$ is coterminal with $A - B$, and the area of the triangle OQP is $\frac{1}{2}\sin(A - B)$ in all cases, the area being regarded as positive if the triangle is described in a counter-clockwise sense and otherwise negative.

The area of a triangle with vertices $(0, 0)$ (x_1, y_1) (x_2, y_2) is

$$\frac{1}{2}\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix},$$

the same convention as to sign being adopted as above. (The determinant notation is not essential.)

Thus $\qquad 2\triangle OQP = \sin(A-B)$

$$= \begin{vmatrix} \cos B, & \sin B \\ \cos A, & \sin A \end{vmatrix}$$

$$= \sin A \cos B - \cos A \sin B.$$

The formulae for $\sin(A+B)$ and $\cos(A \pm B)$ follow in the usual way.

RAYMOND SMART.

1451. *A note on magic squares.*

All multiplicative magic squares of order three are given by the formula :

Kab	$Kb^2\theta^2$	$Ka^2\theta$
$Ka^2\theta^2$	$Kab\theta$	Kb^2
$Kb^2\theta$	Ka^2	$Kab\theta^2$

a, b, θ, K are integers, a, b relatively prime, $\theta > 1$. Product of all rows, columns, and diagonals is $K^3a^3b^3\theta^3$.

ERIC GOODSTEIN.

1452. *Solution of quadratic equations by the slide rule.*

The slide rule usually consists of four scales, which we may designate (from top to bottom) by A, B, C, D respectively, the scales A and D being fixed, and B and C movable. The identical scales A and B consist of numbers which are the squares of the corresponding numbers on the identical scales C and D.

By dividing through by the coefficient of x^2, a quadratic equation can always be reduced to the form $x^2 + ax = b$. In this form it can sometimes be very conveniently solved by the slide rule.

Take for example the equation $x^2 + 1{\cdot}8x = 12$. It will have one positive and one negative root. In order to obtain the positive root it is necessary to find a number whose square, together with 1·8 times the number, makes up 12. Move the slide of the rule to the left until 1·8 on the scale C coincides with 1 on the scale D. Then if x be any number on D, the coinciding number on C will be $1{\cdot}8x$, and the coinciding number on A will be x^2. Hence the sum of the coinciding numbers on A and C will be $x^2 + 1{\cdot}8x$. Move the cursor along until this sum is 12, then the number on D gives us the required value of x, viz. 2·68.

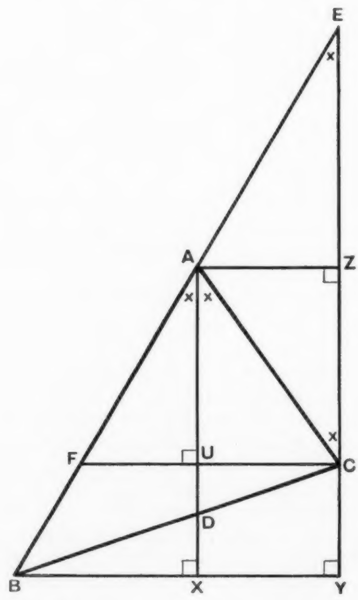
In equations such as the above, it is usually only the positive root that is required in practice ; but the negative root, if necessary, can also be found by this method. For the negative root of the equation $x^2+1{\cdot}8x=12$ is the positive root, with its sign changed, of $x^2-1{\cdot}8x=12$. This positive root may be found thus. Keeping the slide still in the same position, now *subtract* the number on the C scale from the coinciding number on the A scale, and move the cursor until this difference is 12. We thus find that $x^2-1{\cdot}8x=12$ when $x=4{\cdot}48$. Hence the negative root of $x^2+1{\cdot}8x=12$ is $-4{\cdot}48$.

Thus the roots of $x^2+1{\cdot}8x=12$ are $2{\cdot}68$ and $-4{\cdot}48$, and the roots of $x^2-1{\cdot}8x=12$ are $4{\cdot}48$ and $-2{\cdot}68$. W. H. SALMON.

1453. *The half-angle formulae and the tangent rule.*

Let AD bisect $\angle A$; CE be parallel to AD ; AZ and BXY be perpendicular to CE. Then $\angle BEC$ is necessarily acute, so that

$$BC^2=BE^2+CE^2-2CE\,.\,YE\,;$$



hence
$$\begin{aligned}BE^2-CB^2&=2CE\,.\,YE-CE^2\\&=CE(2YE-CE)\\&=2EZ\,.\,2YZ\\&=4EZ\,.\,AX\end{aligned}$$

or $$(b+c)^2-a^2=4b\cos\tfrac{1}{2}A\,.\,c\cos\tfrac{1}{2}A,$$
whence $$\cos\tfrac{1}{2}A=\surd\{s(s-a)/bc\}.$$

If CF is perpendicular to AD, $\angle BCF$ is necessarily acute, and
$$BF^2=CF^2+CB^2-2CF\,.\,BY$$
or
$$\begin{aligned}CB^2-BF^2&=2CF\,.\,BY-CF^2\\&=CF(2BY-CF)\\&=2CU\,.\,2BX,\end{aligned}$$
so that $$a^2-(c-b)^2=4bc\sin^2\tfrac{1}{2}A,$$
whence $$\sin\tfrac{1}{2}A=\surd\{(s-b)(s-c)/bc\}.$$
Also
$$\begin{aligned}\angle CBY&=\angle BCF\\&=C-(90^\circ-\tfrac{1}{2}A)\\&=\tfrac{1}{2}(C-B).\end{aligned}$$
Thus
$$\begin{aligned}\tan\tfrac{1}{2}(C-B)&=CY/BY\\&=(AX-AU)/(BX+CU)\\&=\frac{c\cos\frac{1}{2}A-b\cos\frac{1}{2}A}{c\sin\frac{1}{2}A+b\sin\frac{1}{2}A}\\&=\frac{c-b}{c+b}\cdot\cot\tfrac{1}{2}A.\end{aligned}$$

B. A. SWINDEN.

1454. *An array of squares.*

13	12	15	4
10	5	3	7
14	1	11	6
2	9	8	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The *Fifteen Puzzle*, which enjoyed a brief but remarkable popularity about fifty years ago, consisted of 15 numbered cubes or squares in a box which could hold 16 ; and the idea was to use the blank square to move the pieces until they were arranged in order of magnitude. It was pointed out by F. J. W. Whipple in the *Math. Gazette*, Vol. XIII, p. 126, Note 837, that with any arrangement it was an even chance that the puzzle was insoluble, and he gave a rule to determine whether it was so or not.

In proving this rule or an equivalent form of it, we may consider more generally any connected array of squares in which only one

square is left blank. We first count the number of inversions, an inversion being counted when a number comes in front of a smaller one, the natural order being taken as that in the final arrangement. Thus in the Fifteen Puzzle above the number of inversions is

$$(12+11+12+3)+(8+3+2+3)+(6+4+1)+1=66.$$

Actually only the parity of this sum is needed, and this can be determined quickly by crossing out all the even numbers and pairs of odd ones.

Consider how the inversions are affected by moving the squares. Obviously sideways shifts make no difference. An upward or downward move of a number to a blank square changes its situation relative to the m numbers, say r greater and $m-r$ smaller, between its two positions, and so alters the number of inversions by

$$r-(m-r)=2r-m,$$

which is even or odd according as m is even or odd. In the Fifteen Puzzle $m=3$, and each upward or downward shift counts for an odd number of inversions. For an array of 5×5 squares with numbers 1 to 24, $m=4$, and for this or for any rectangular array with an odd number of columns the parity of the number of inversions can never be changed.

In the general case when m may be odd for certain rows, we need only consider the net number of rows which each number has to pass over to reach its final position, an up and a down move from the same row cancelling. The sum of these numbers for the whole array is always even. For we can get to any position to any other position in which the blank square is in the same place by interchanging pairs of numbers. Take the positions in which just two numbers are interchanged; the difference between the sum of the total upward shifts is either zero or twice the number of rows between the two numbers, and so is even. Hence whatever the array the number of inversions affected in moving from its initial to its final position is always even, and therefore if the number of inversions in the original array is odd, the puzzle is insoluble. This supposes that the blank square is in the same place in both cases, and Dr. Whipple's rule can therefore be applied. In the Fifteen Puzzle if the blank square is in the first or third row at the start, the total number of inversions must be odd for the puzzle to be soluble.

We have proved this condition to be necessary; but supposing it satisfied, it is easy to formulate a way of solving the puzzle. It consists in moving squares along the sides of a rectangle and using a set of 2×2 squares, one of which is the blank, connected with it, for the purpose of interchange. If we call such a rectangle + square a *circuit*, it is necessary for a given array to be soluble that every square shall be on the side of a circuit. This, with the condition about the inversions, form necessary and sufficient conditions for solution. An array of 4×4 squares with one extra square and with numbers 1 to 16 is obviously an unsuitable array and can only

exceptionally be solved. But with two extra squares at the same end of adjacent rows a circuit can be made with every square.

The minimum total number of moves (sideways included) required to rearrange the squares is an interesting problem. The Fifteen Puzzle only requires about 150 or less. More difficult variants can be obtained by filling in certain of the 16 squares with blanks. Thus in the appended diagram, the number of inversions (24) is even and

4	7	9	1
2			10
8	3		6
11	12	5	

every square is on a circuit ; thus the problem is soluble. But it requires about 900 moves to rearrange the numbers in order of magnitude. H. V. MALLISON.

1455. *A note on Sylvester's method of elimination.*

Let

$$f(x) \equiv a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4,$$
$$\phi(x) \equiv b_0x^3 + b_1x^2 + b_2x + b_3 \equiv b_0(x-\alpha)(x-\beta)(x-\gamma),$$

$$D \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \end{vmatrix}, \qquad \Delta \equiv \begin{vmatrix} \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^6 & \beta^5 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^6 & \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

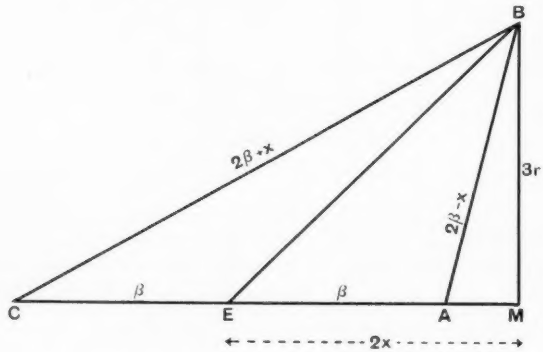
Sylvester's theorem is that $D=0$ is the necessary and sufficient condition that f and ϕ should have a linear factor in common, and similarly for any two polynomials. In fact, $D=0$ is the result of multiplying f by x^2, x, 1 and ϕ by 1, x, x^2, x^3 and eliminating $x, x^2, x^3, x^4, x^5, x^6$. But, as pointed out by M. Bôcher, *Introduction to Higher Algebra* (Macmillan, 1907), p. 200, this only proves that $D=0$ is a necessary condition. It is the purpose of this note to give an easier proof than Bôcher's of the sufficiency of the condition. All we have to do is to form the product ΔD by the ordinary rule for multiplying two determinants, when D is at once seen to be $b_0{}^4 f(\alpha) f(\beta) f(\gamma)$. It has been assumed that no two of α, β, γ are equal. If, for example, $\alpha=\beta=\gamma$, replace the second and third rows

of Δ by the first and second derivatives of the first row with respect to α.

The above method can be adapted to find the necessary and sufficient conditions that f and ϕ may have two or three linear factors in common. But the inductive method outlined by Bôcher, *loc. cit.*, p. 197, is simpler. HAROLD SIMPSON.

1456. *A useful triangle.*

The triangle ABC constructed from the following specifications is often useful for purposes of illustration.



$$a=2\beta+x, \qquad s-a=\beta-x,$$
$$b=2\beta, \qquad s-b=\beta,$$
$$c=2\beta-x, \qquad s-c=\beta+x.$$
$$s=3\beta,$$

The area of $ABC=\beta\surd\{3(\beta^2-x^2)\}=3\beta r$, where r is the radius of the inscribed circle, so that $3r^2=\beta^2-x^2$.

The triangle is obtuse-angled, right-angled or acute-angled as $x \gtreqqless \frac{1}{2}\beta$.

$$\tan\tfrac{1}{2}A=\surd\{(\beta+x)/3(\beta-x)\},$$
$$\tan\tfrac{1}{2}C=\surd\{(\beta-x)/3(\beta+x)\},$$
$$\tan\tfrac{1}{2}B=r/\beta\,;$$
$$\tan\tfrac{1}{2}A\,\tan\tfrac{1}{2}C=\tfrac{1}{3}\,;$$
$$\cos A=(\beta-2x)/(2\beta-x), \quad \cos C=(\beta+2x)/(2\beta+x).$$

The median to the base 2β is $BE=m$, where

$$m^2=3\beta^2+x^2.$$

The projection EM of this median on the base is $2x$.

If R is the radius of the circumscribed circle,

$$6rR = 4\beta^2 - x^2.$$

If Q is the point of contact of the inscribed circle with CA, then $EQ = x$.

I. M. MATHEWS.

1457. *On Note* 1429.

In Note 1429, Dr. R. F. Muirhead showed how to construct rational functions of the form $(ax^2+2bx+c)/(a'x^2+2b'x+c')$ with given maximum and minimum values. In devising such problems, I have used a more direct method which can be explained by a particular example:

To determine the rational function

$$y = (ax^2+2bx+c)/(a'x^2+2b'x+c'), \qquad \text{.................(1)}$$

with maximum and minimum values 5 and 9, not necessarily respectively.

(1) is equivalent to

$$(a'y-a)x^2+2(b'y-b)x+c'y-c=0,$$

the discriminant of which is

$$(b'y-b)^2-(a'y-a)(c'y-c)$$

or

$$(b'^2-a'c')(y-5)(y-9).$$

Hence y lies inside or outside the range (5, 9) according as

$$b'^2-a'c' \lesseqgtr 0.$$

Since the discriminant vanishes at the critical values 5 and 9, the constants a, b, c, a', b', c' must be chosen to satisfy

$$(5b'-b)^2 = (5a'-a)(5c'-c),$$
$$(9b'-b)^2 = (9a'-a)(9c'-c).$$

To obtain rational solutions, we give b, b' such values that $(5b'-b)^2$ and $(9b'-b)^2$ have many factors. For example, if

$$b=3, \quad b'=1,$$

we have the following scheme:

	$5a'-a=$	1,	-1,	2,	-2,	4,	-4, ...
	$9a'-a=$	2,	1,	12,	2,	-4,	-6, ...
	$5c'-c=$	4,	-4,	2,	-2,	1,	-1, ...
	$9c'-c=$	18,	36,	3,	18,	-9,	-6, ...
whence	$a=$	1/4,	7/2,	21/2,	7,	-14,	3/2, ...
	$a'=$	1/4,	1/2,	5/2,	1,	-2,	$-1/2$, ...
	$c=$	27/2,	54,	$-3/4$,	27,	$-27/2$,	$-21/4$, ...
	$c'=$	7/2,	10,	1/4,	5,	$-5/2$,	$-5/4$, ...

and

$b'^2-a'c'$ is	+,	−,	+,	−,	−,	+,	...

and so on, extended indefinitely.

Thus the functions

$$(x^2+24x+54)/(x^2+8x+14),$$
$$(42x^2+24x-3)/(10x^2+8x+1),$$
$$(21-24x-6x^2)/(5-8x+2x^2),$$

cannot take values between 5 and 9, while the functions

$$(7x^2+12x+108)/(x^2+4x+20),$$
$$(7x^2+6x+27)/(x^2+2x+5),$$
$$(28x^2-12x+27)/(4x^2-4x+5),$$

can only take values between 5 and 9.

If b and b' be suitably chosen, there is almost no limit to the number of such functions of the type (1) that can be constructed with given maximum and minimum values. In the above case, for instance, without using fractional values in the first four rows of the table, 108 functions can be constructed, although in some cases a' or c' may be zero. R. HOLMES.

1458. *On Note* 1407.

In Note 1407 Mr. A. G. Granston Richards described a model for exhibiting the values of the sine and cosine of an angle. Mr. William J. Hazard, of the University of Colorado, writes to say that a similar model was constructed by him in 1924 and has been used regularly ever since then. It has been copied by a number of teachers in Colorado, and a description of it was given in the *American Mathematical Monthly*, XXXVI, No. 1, January 1929.

1459. *On Note* 1430.

It has been pointed out that the proof of the binomial theorem for a positive integral exponent given in Note 1430 (December, 1939), is essentially the same as that given by Mr. Neil McArthur in the *Journal of the Glasgow Mathematical Association*, Vol. I, No. 2, December 1937.

CORRECTION.

In Note 1418, October 1939, pp. 402-404, *for* " Montfort " *read* " Montmort ".

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be below the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. The names of those sending the questions will not be published.

The Secretary would be glad to receive any solutions that have not yet been returned.

REVIEWS.

James Gregory. Tercentenary Memorial Volume. Edited by H. W. Turnbull. Pp. viii, 524; 5 plates. 25s. 1939. (Published for the Royal Society of Edinburgh by G. Bell & Sons)

Late in July 1914 the Royal Society of Edinburgh and the Edinburgh Mathematical Society held an International Congress to honour the memory of John Napier, who is best known through his invention of logarithms. The lectures then given were published in the Napier Memorial Volume in 1915. In the preface to this volume the editor—the late Dr. C. G. Knott—refers to the meetings of the congress being attended " by men and women whose nationalities were fated to be in the grip of war before a week had passed ". In 1938 the same two societies held a colloquium at which was celebrated the tercentenary of the birth of James Gregory, who was professor of Mathematics at the University of St. Andrews from 1668 to 1674, and thereafter occupied a similar position in Edinburgh until his death in 1675. The Gregory Memorial Volume was published in October last under somewhat similar circumstances to those that prevailed in 1915, but this alone seems to be the only obvious point that the two works have in common. The congress in 1914 was for the sole purpose of honouring Napier, whose name was well known to all mathematicians, and even to those whose scientific knowledge was perhaps only slight. It is true that the Napier Volume showed that Napier was interested in branches of mathematics other than those directly connected with logarithms, but the papers and addresses were given by more than twenty different lecturers, who were all presumably enthusiastic as to the worthiness of Napier's work being thus recorded for the benefit of the mathematical world. In contrast to this, the colloquium of 1938 only incidentally celebrated the tercentenary of Gregory's birth, most of the papers being on topics of modern mathematical research, including one very interesting series on Mathematical Biology. In other words, Napier was well known even *before* the congress in 1914, while Gregory was very little known even *after* the colloquium in 1938. The Napier Volume was the collected tributes of a number of admirers; the Gregory Volume is very largely due to one man, who has laboured patiently and enthusiastically for almost nine years in a field where mathematical prowess is only one of the many talents required for the successful completion of the work. Professor Turnbull deserves congratulation from the scientific world for this book, which not only establishes the greatness of his ancestor in the chair of Mathematics at St. Andrews, but is also an admirable survey of the state of science in the seventeenth century. With a generosity that is characteristic, Professor Turnbull states that the volume is the result of the labours of a committee, but while its compilation doubtless owes something to those whose names are mentioned in the preface, yet it is largely a Turnbull production, and will be regarded as such by his friends and acquaintances. In the third section of the book there are six short essays, four of which are not by the Editor, and while these are very interesting in themselves, yet they do not contribute to the main theme.

Professor Turnbull's thesis is that James Gregory discovered the infinitesimal calculus quite independently of Newton—or any other scientist—and that Gregory made such progress in this branch of mathematics that he was able to use Taylor and Maclaurin expansions with facility. Not only did Gregory thus establish a claim to be numbered among the great, but he also did valuable work in the theory of equations and in the theory of numbers. He was considered by his contemporaries as being second only to Newton. To state that Gregory used Taylor's expansions about forty years before these were invented

is so serious that it could only be made if there was ample evidence in support of it. The independent discovery of the calculus is not quite so startling, because as Professor Turnbull says, " it was in the air ", but even so there must be available substantial documentary proof that Gregory deserves to be placed with Newton and Leibniz. To show that his statements are not without foundation, Professor Turnbull gives Gregory's correspondence during the years 1667 to 1675. This he translates from the original Latin, and then proceeds to show how Gregory arrived at the results given in his letters by transforming the mathematical processes of the time into those of the present day. Even that field that the Editor has made his own—The Theory of Matrices—is used to explain one of the complicated formulae that is given.

Most of Gregory's correspondence was with John Collins, a clerk employed in various government offices in London, and who acted as a connecting link between the newly founded Royal Society and scientists abroad, which term at this time included Scotland. Collins plays such an important part in the story that he deserves further mention. He was a minister's son and was originally apprenticed to a bookseller; this he forsook for the sea, and after seven years abroad he returned home to become clerk to a number of farmers who exported foodstuffs; it was then he started corresponding with foreign scientists. Although his own education had been but scanty, he taught himself a great deal and wrote several important commercial treatises. At the time of his first letter to Gregory—1667—he was an accountant at the Excise Office, Bloomsbury, and thereafter until his death in 1683 at the age of 59, he was in the employ of the government. In his later years he was frequently consulted in the costing of public works, and also he was of great assistance to many scientists in arranging for the publication of their books and papers. Wallis and Barrow were indebted to him in this way, for the Plague and the Fire of London had made printing and publication very difficult and costly. Consequently although Collins himself was not a scientist yet through him many important works were published, and also scientists in England and Scotland were kept in touch with their confreres in Europe. Thus he has been worthily called the Mersenne of his time. The correspondence shows him to be a man of shrewd judgment well able to assess the value of a scientific work, but he seems to have been most keenly interested in the theory of equations and he keeps on returning to this topic again and again.

The attitude of the scientists of the period towards each other seems to have been peculiar, for the general impression given here is that in general they did not trust each other's integrity. They were only too ready to accuse a contemporary of appropriating results without acknowledgment to the original discoverers, and to find fault rather than to give an unbiassed opinion. Small wonder that in this atmosphere of suspicion and mutual mistrust Newton—for example—for many years published only his lectures at Cambridge, and this because he was compelled to do so under the terms of his appointment at the University. He disliked controversy and he knew that every scientific paper published would probably arouse a storm somewhere. Pell was another mathematician who kept his knowledge very much to himself, and his work even now is not known. Collins's letters contain many references to him and to results which he had obtained but kept shrouded in mystery. His manuscripts are in the British Museum and will probably provide a fertile field of research for some student in the future. One is forced to the conclusion that science was in the adolescent stage during the seventeenth century, and that the quarrels and bickerings that took place were the natural expression of men who were groping their way to scientific maturity. Gregory himself early

in his career fell foul of Huygens, then Anderson was severely criticised by Gregory, and later Leibniz was reviled by the Royal Society, the latter even going to the length of publishing the *Commercium Epistolicum* in support of Newton's prior discovery of the calculus. There is no doubt that this suspicion and mistrust was partly due also to the difficulties of inter-communication. Even in Britain these were against the free exchange of ideas of a scientist in London and one in Scotland. There was only one carrier each week from St. Andrews to Edinburgh, from whence packages addressed to London either went by road or by sea. At Newcastle everything going to Scotland was subject to a customs duty, the imposition of which caused delay and increased the expense. Postage from London to St. Andrews was fivepence for a single sheet of paper, which caused Gregory to cram his sheets full in order to convey a message as cheaply as possible. This has made Professor Turnbull's task all the more difficult, judging from the specimens of Gregory's manuscripts that are given in the plates.

The greater part of the task was the examination of Gregory's correspondence, some of which was published in 1841 by Rigaud in the " Correspondence of Scientific Men of the Seventeenth Century ". The remainder of the letters are to be found in the collections of manuscripts in the libraries of the Universities of Edinburgh and St. Andrews, and in the collection of manuscripts owned by the Royal Society. At the time paper was expensive and Gregory was remarkably economical, for he used the blank spaces on letters sent to him for rough notes of all kinds. It is these rough notes which are of fundamental importance in establishing the claim that Turnbull makes for his predecessor. It is certainly through them that he is able to show that Gregory used the Taylor and Maclaurin expansions, and it appears that David Gregory—James' nephew and successor at Edinburgh—attested that fair copies of these notes were made, but these have not yet been found.

In my opinion Gregory's best work was done in Mathematical Analysis. Here he shows a vigour and a command of the subject which many modern students might envy. He is rigorous in his use of expansions and shows that he knows the meaning and significance of convergence ; incidentally he was the first mathematician to use this term and apply it to series. The book provides such a rich store of Gregory's mathematical powers that it is only possible to give two examples of them.

In a letter written to Collins at St. Andrews on 15th February, 1671, Gregory gave seven series, six of which are examples of Taylor's Theorem. The seven series, in order, are the expansions of $\tan^{-1} x$, $\tan x$, $\sec x$, $\log \sec x$, $\log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, $\sec^{-1}(e^x\sqrt{2})$, and $2\tan^{-1}\left(\tanh\frac{x}{2}\right)$, each in ascending powers of x. The second, third, sixth, and seventh series were obtained by successive differentiation, the fourth and fifth by integrating the third and second. The first—now known as Gregory's series—was probably obtained by integrating the power series for $(1+x^2)^{-1}$. Naturally Gregory did not arrive at these results without a great deal of thought and hard work, and again he worked out these results on any odd scraps that were to hand before communicating them to Collins. Fortunately Turnbull has discovered the rough notes for this particular letter in the blank spaces left on a letter from an Edinburgh bookseller—Gideon Shaw—to Gregory. These notes commence with some false starts, and then if m is the ordinate and t the subtangent of a curve (so that $m:t$ denotes the gradient of the tangent), Gregory calculates successive quantities m, each accompanied by the corresponding t, and thus each successive m is the derivative of the previous, *i.e.*

$$m=y,\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2}\ldots,$$

$$t=y\Big/\frac{dy}{dx},\quad \frac{dy}{dx}\Big/\frac{d^2y}{dx^2},\quad \frac{d^2y}{dx^2}\Big/\frac{d^3y}{dx^3}\ldots.$$

On this double sheet from Gideon Shaw there are sixteen mathematical items which show Gregory was working on the principle of successive differentiation. He calculated the seventh or eighth derivatives of the seven functions mentioned above, and in the seventh derivative of $\tan x$ he made an arithmetical error which naturally is passed on in the calculation of the eighth derivative. Now this arithmetical error appears in the letter of which these notes were the rough draft, and this is the chief link between the letter and the notes. There is also the common notation of the two, and the undisputable fact that to develop the seven functions in Taylor or Maclaurin series their derivatives are required. If then Gregory did not know the theorem

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots$$

two questions must be answered; first, how did he find these series given in the letter; and, second, for what purpose did he calculate these derivatives? It is not claimed that he explicitly stated Taylor's Theorem, but it is claimed that he was familiar enough with it to use it in order to obtain these series. The uncertainties at the beginning of the notes show that Gregory was on unfamiliar ground, and that he was literally groping his way to a theorem which did not appear until 44 years later. These notes are worth considerable further scrutiny by mathematicians, if only to show that this claim of Professor Turnbull's is not made in the first flush of enthusiasm, but as the result of a careful enquiry in which each piece of evidence is fully investigated. It is not possible within the scope of this review to give more than the gist of the arguments, but it is hoped that enough has been given to show that—at least—this letter with the relevant rough notes and the step-by-step explanation of Professor Turnbull are worth the close attention of all interested in the development of Mathematics.

The second example that I have chosen is Gregory's solution of Kepler's Problem, *i.e.* to find a straight line through a given point on the diameter of a semicircle, so that it divides the area of the semicircle in a given ratio. Wren solved it in terms of a cycloid and Wallis quoted this in his book on that curve. Here again Gregory gave the bare skeleton of the solution of the problem in a letter to Collins written at St. Andrews on 9th April, 1672, the rough notes for which are on the Gideon Shaw letter. At the time Gregory seems to have been interested in the cycloid, an interest which was doubtless aroused by reading *De Cycloide*, by Wallis; in fact in his notes he refers to one of the diagrams in the book. Gregory's solution amounts to finding the coordinates of a point (x, y) in the form $x=r(1-\cos\phi)+b\phi$, $y=r\sin\phi$, and then calculating $\frac{dy}{dx}$ and the next four successive derivatives in terms of ϕ. At $\phi=0$ these are

$$\frac{r}{b},\quad -\frac{r^2}{b^3},\quad -\frac{r}{b^3}+\frac{3r^3}{b^5},\quad \frac{7r^2}{b^5}-\frac{15r^4}{b^7},\quad \frac{r}{b}-\frac{60r^3}{b^7}+\frac{105r^5}{b^9}.$$

Then by Maclaurin's Theorem the value of y is

$$y=\frac{rx}{b}-\frac{r^2x^2}{2b^3}+\frac{r^3x^3}{2b^5}-\frac{rx^3}{6b^3}+\frac{7r^2x^4}{24b^5}-\frac{5r^4x^4}{8b^7}+\frac{7r^5x^5}{8b^9}-\frac{r^3x^5}{2b^7}+\frac{rx^5}{120b^5}\ldots,$$

which is precisely Gregory's series for the general cycloid. When we write $b=r$, we obtain the series for the ordinary cycloid. Professor Turnbull refers to this as "Gregory's *tour de force*", but we might well regard the disentanglement of it as being "Turnbull's *tour de force*", for in the rough notes Gregory refers to the "tangent", which when taken to mean the derivative does not make sense in the series that follows. The present reviewer made an abortive attempt some years ago to find an explanation of Gregory's solution of Kepler's Problem, and it was this stumbling block that he failed to surmount. Professor Turnbull however takes the term "tangent" to mean "subtangent", because Gregory seems to have used the subtangent where the modern student would consider the tangent. With this substitution, the solution can be followed. There are a few errors, but many are of the "slip" variety that in a work of this magnitude are almost inevitable. On page 9, instead of the meeting between Gregory's son and his kinsman—Rob Roy—being described in *Old Mortality*, it is in *Rob Roy*. In section (xvi), page 356, $a=r\log\theta$, and not $a=r\theta$; the energy equation on page 371 has a numerical factor "24" instead of "2". The explanation to the matrix on page 460 seems to be faulty, and the line "In each row of four elements they form a proportion $M:a::C:D$, etc." is not correct. Instead it should read "$P=\sqrt{L(2L+M)}$, and a^2 is the difference between $4L^2$ and M^2, Q^2 is the difference between $4L^2$ and B^2". Consequently the following line should then be $M:2L::C:B$. On page 462 occurs the phrase "triplicates the true notes", which requires further explanation. By "the true notes" Gregory means the "exact formula", then if ϕ is some formula to which ψ is an approximation, $\psi-\phi$ triplicates the true notes of ϕ when $\psi-\phi=O(\phi^3)$; it will n-plicate the true notes when $\psi-\phi=O(\phi^n)$. Thus $|\psi-\phi|$ is of the same order as the nth power of the exact formula. Gregory as a rule worked with areas and therefore his ϕ is proportional to the square of the sine or tangent.

It is to be hoped that this book will be given the close attention that it deserves, if only for the fact that it shows very clearly how very full of scientific discovery was the seventeenth century. The well-known names all appear—Newton, Huygens, Hooke, Flamsteed, Tschirnhaus—and also those of others whose contributions to the sum total of scientific knowledge have either been forgotten, or what is more likely, have never been known. It reveals the poverty of our scientific education in that most of us—mathematicians, scientists, or merely teachers—know little or nothing of the pioneers in our subjects, of the difficulties they had to surmount; difficulties which were sometimes due to the foolish prejudice of their contemporaries and sometimes due to their own ignorance. This latter they could—in time—turn into knowledge, but the obstacles put in their paths by others provided much harder nuts to crack. For example, Gregory's last year at St. Andrews was made most unpleasant by the attitude of the University authorities to mathematics, so that he was very pleased to accept the chair at Edinburgh. Consequently it is a mark of real progress in our general mathematical culture that a mathematician of the standing and attainments of Professor Turnbull should be willing to use his leisure in investigating a part of the history of the subject, and also that his research should be embodied in a volume produced under the auspices of two learned societies. This Memorial Volume is thus noteworthy in that many for the first time will hear of James Gregory, who through the evidence it gives now takes his place among the greatest mathematicians, a place to which he is entitled by his mathematical genius, but to which he would not have been elevated had it not been for the enthusiasm, patience and skill of one who now occupies with distinction the chair that he once held.

ALEX. INGLIS.

Topological Groups. By L. PONTRJAGIN. Translated by E. LEHMER. Pp. ix, 299. 22s. 6d. 1939. Princeton Mathematical Series, 2. (Princeton University Press ; Humphrey Milford)

This book presents some of the most interesting work on groups which has been done during the last ten years. It opens with three fairly elementary chapters, on the generalities of group theory, on topological spaces and on topological groups. The next chapter describes the linear representations of compact groups and the theory, due, in the form given here, to J. von Neumann, of abstract " invariant integrals " on a group. Such an integral is a functional, depending on an arbitrary continuous function $f(x)$, which is defined over the group. It satisfies certain conditions in virtue of which it may reasonably be described as the " integral " of $f(x)$ over the group, by analogy with the case of a compact Lie group, when it actually is this integral, and is invariant in the sense that it is unaltered when $f(x)$ is replaced by $f(ax)$, $f(xa)$ or by $f(x^{-1})$, where a is any constant element in the group and x^{-1} is the inverse of x. A theory of integral equations is developed, which leads to the following theorem on the linear representations of the group.

If one representation is chosen from each class of mutually equivalent representations, then the elements of the matrices which constitute these representations form a uniformly complete set of functions on the group (a set of functions is uniformly complete if, and only if, a suitable linear combination of them, with constant coefficients of which all but a finite number are zero, is an arbitrarily close approximation to a given continuous function on the group). As a corollary to this theorem it follows that, given an element a of the group, other than the unit element, there is a representation in which a corresponds to some matrix which is not the unit matrix. This does not mean that there is any single " true " representation, meaning one in which different elements correspond to different matrices. If there is such a representation, the group, being compact and isomorphic to a sub-group of a Lie group, is itself a Lie group. The next chapter presents the theory which was developed by Pontrjagin himself in connection with Alexander's duality theorem. By means of this algebraic theory Pontrjagin was able to extend the latter from a statement concerning a topological image, in Euclidean n-space R^n, of a finite polyhedron, to a theorem about an arbitrary compact sub-set, F, of R^n. For example, if real numbers, reduced mod. 1, are taken as coefficients in forming cycles, then the homology groups of F have a topology, and are compact, commutative groups. With these coefficients for cycles in F, the duality theorem takes the form that the (compact) kth homology group of F is orthogonal to the (discrete) $(n-k-1)$th homology group of R^n-F, with integral coefficients. The general situation is that, to each locally compact group G, which satisfies the second axiom of countability, corresponds an " orthogonal " group X, which is of the same type, and G is orthogonal to X. Each of X and G is uniquely determined by the other, up to isomorphism. In fact each is the " character group " of the other, the character group of G being the set of homomorphisms of G in the additive group of real numbers, reduced mod. 1. If G is compact, then X is discrete and vice versa. This theory is rich in theorems which are of first class importance both for topology and for algebra. I state one of them, chosen more or less at random. *A compact, locally connected and connected commutative group is the direct sum of a finite or countable number of sub-groups, each of which is isomorphic to the additive group of real numbers, reduced mod.* 1.

The chapter concludes with a brief account of topological fields, including the theorem that the real and complex fields and the quaternions are the only locally compact, connected fields, which satisfy the second axiom of counta-

bility. There follow three short chapters on Lie groups, on compact groups and on locally isomorphic groups. The second of these contains von Neumann's solution, for compact groups, of the famous problem, proposed by Hilbert, which is to prove that any " locally Euclidean " topological group is a Lie group. The book concludes with a fairly complete account of the Lie theory. This does not include the deep algebraic analysis of an infinitesimal group, or Lie algebra as it is generally called nowadays, which was originated by Killing and carried out by Cartan. But it goes beyond the elementary formalism of the subject in that the distinction between a complete group and a "local" group is carefully drawn. Among other things the third fundamental theorem of Lie is proved in its complete form, which states that there is a complete group, not merely a local group, for which a given set of constants are the constants of structure, subject to the necessary algebraic relations between them.

The book is written with remarkable clarity and the translation is so good that little, if anything, can have been lost. As a by-product it provides a first class introduction to group theory, abstract topology and the Lie theory. But the theorems for which the book is chiefly remarkable are presented so clearly that most of them are easily accessible to any one with a certain amount of mathematical common sense and a taste for abstract thought.

J. H. C. W.

The Mathematical Theory of Huygens' Principle. By B. B. Baker and E. T. Copson. Pp. vii, 155. 12s. 6d. 1939. (Oxford)

This learned and difficult book, inspired by the lectures and influence of Professor E. T. Whittaker, is intended by its authors to be the first of a series of monographs, each complete in itself, on various special topics in the theory of the partial differential equations of mathematical physics. The subjects treated in this volume (and to be treated in later volumes) are chosen as being among those not yet adequately dealt with in existing treatises on the general theory.

Elementary students of physics first come across Huygens' principle in their course on light, and probably few, even among advanced students, learn very much about it. The principle was formulated at a time when light was regarded as a wave-disturbance similar to that of sound in air, although the phenomenon of polarization, which shows that light propagation is essentially more complicated than that of sound, was discovered by Huygens himself. Even in the simplest geometrical form of the principle, however, and as applied to sound, it is not satisfactory without additional assumptions : Huygens had to assume that his secondary waves had effect only where they touched their envelope : further, since there may be an envelope to the rear of an advancing system of waves, as well as in front, it was necessary to assume that only one sheet of the envelope is to be considered. To avoid making this last assumption, it is necessary to give up the purely geometrical theory and resort to analysis.

Two-thirds of the book are devoted to the discussion of Huygens' principle in relation to " scalar " waves of the same nature as sound. Fresnel made an important extension of Huygens' principle by the substitution of purely periodic trains of spherical waves for Huygens' isolated secondary spherical waves, and by using the principle of interference. In this way he was able to account not only for the rectilinear propagation of light of very short wavelength, and the laws of reflection and refraction, but also for certain diffraction phenomena. The second of the four chapters of the book is devoted to the theory of diffraction of scalar waves. The names that figure in these first two

chapters are mainly those of generations earlier than ours, such as Stokes, Helmholtz, Kirchhoff, Weber and, of our own older contemporaries, Volterra and Hadamard.

Even this relatively simple field of application of Huygens' principle is rather formidable, although the quotation "Volterra's proof of the more general theorem just enunciated and a later proof due to Hadamard are both too difficult to give here" indicates that still more severe discussions await him who would penetrate further. That progress is being made is, however, indicated by another quotation, "quite recently Professor Marcel Riesz has discovered an elegant and simple method for dealing with problems of this type . . . ", and by reference to recent work by Kottler and Matthison.

The last third of the book consists of two chapters on Huygens' principle for electromagnetic waves, and on Sommerfeld's theory of diffraction by a perfectly reflecting screen (two dimensional propagation only). Here we meet with work by Larmor, Tedone, and Voigt, as well as of other authors already named: the secondary wave-systems considered by Larmor and Kirchhoff (involving very interesting mathematics) are unfortunately subject to the objection that they do not satisfy Maxwell's equations, so that they are not completely suitable for the solution of diffraction problems. In such problems, as the authors point out, there are also difficulties in the correct physical statement of the boundary conditions. The keynote of the book (the existence of important theoretical problems posed and already vigorously attacked, but as yet far from solved) is maintained to the end, as may be seen from the last paragraph of the book:

"The real difficulty lies in the fact that the black screen is an idealization which cannot be attained experimentally and which has no precise definition in electromagnetic theory. What is needed is a rigorous theory, which does not assume that the screen is perfectly reflecting but takes into account the properties of the material of the screen. An empirical way of finding the effect due to a screen which is neither perfectly black nor perfectly reflecting is to suppose that the reflected wave has its amplitude and phase reduced in some definite way, due to the imperfections of the reflecting power of the screen. This amounts to multiplying the term representing the reflected wave in Sommerfeld's solution by a complex constant. The effect of this modification of Sommerfeld's theory has been worked out in some detail by Raman and Krishnan."

The book is thus as far removed as possible from most of the mathematical treatises seen by school pupils and even by university students, in which all is well worn, cut and dried. Its rather severe character will limit its appeal, but it will give to its readers a sense of the untiring and continually renewed attacks, by a restricted mathematical *élite*, on the unsolved problems met with almost at the outset of the electromagnetic theory of light: it may be hoped and expected that it will also stimulate some among its readers to carry on this attack. S. C.

Elementary Calculus. By G. W. Caunt. Pp. 388. 7s. 6d. 1939. (Oxford)

The fact that this book is a "smaller model" of the well-known *Introduction to the Infinitesimal Calculus* is a recommendation in itself, and the word "model" is apt in this connection for the volume merits that description. Clear writing and diagrams, a well chosen order of work, and numerous examples combine in the attractiveness.

Much of the larger book is here and occasionally (*e.g.* where an example on tangent and normal now precedes the general case, and again where the Mean Value Theorem is incorporated in the chapter on Taylor's Theorem) a slight

rewriting will be found which tends to improve the value of the work in view of the class of student for which this volume is intended (the book is designed to be " up " to H.S.C. standard).

The chief omissions from the original book are the chapters on Partial Differentiation, Differential Equations and those dealing with the Applications to Mechanics and Physics. One may perhaps regret the latter most, for these applications often give the first indication of how useful, if only as a time and labour saver, the Calculus may be, if taken at a stage when the beauty of the work for its own sake will be unappreciated. However, problems on Areas, Volumes and Moments of Inertia do appear, as the following annotated summary of the Contents shows.

Chapter I. Functions, Graphs (with an appendix on Conic Sections).

Chapter II. Limits, Continuous Functions (very clearly set out with numerous worked examples).

Chapters III, IV, V. Differentiation and Geometrical Applications.

Chapter VI. Maxima and Minima (here the many examples are more stereotyped than in the rest of the book).

Chapter VII. Successive Differentiation.

Chapter VIII. Simple Integration (with examples on areas, and volumes of solids of revolution).

Chapters IX and X. Exponential and Inverse Functions.

Chapters XI and XII. A very orderly and clear arrangement of various methods of integration, and of the properties of definite integrals.

Chapter XIII. Applications including lengths of curves and areas of surfaces.

Chapter XIV. Polar Coordinates (with particularly clear diagrams of a few important curves).

Chapter XV. Centroids, Centres of Pressure, Moments of Inertia.

Chapter XVI. Curvature (including a note on envelopes).

Chapter XVII. Taylor's Theorem (with worked examples which do *not* ignore the " remainder ").

The exercises are often divided into two sections A and B, those in the latter being " for the most part rather more difficult ". Answers are given on perforated sheets at the end of the book—in the review copy rather too well perforated as the pages were found to tear away over readily.

Tables of formulae, common and natural logarithms, Trigonometric ratios, powers, and Exponential and Hyperbolic functions, are included, but there is no index—an omission which might well be amended in future editions. For future editions there certainly should be, as a great demand for this book seems almost inevitable. F. W. K.

An Introduction to the Calculus. By K. R. GUNJIKAR. Pp. xiv, 341. Rs. 4 (Indian). 1938. (Oxford)

In his book on the teaching of Mathematics, Mr. F. W. Westaway says that " the degree of rigour that can be exacted at any stage must necessarily depend on the degree of intellectual development of the pupil ".

Professor Gunjikar presumably recognises the limitations of his pupils, or rather of college students in India generally ; for this book is written for them as a text which is " rigorous in the long run ". Criticism of the standards attained (or unattained) is disarmed by the remark in the preface that " some sections of the book are meant to stimulate interest and to enlighten the curious rather than to serve as a complete logical analysis ".

The result of this attitude is to make the space in this book appear irregularly distributed. Some 50 pages are devoted to " Limits ", with definitions, " precise definitions " and " practical definitions ", but only brief attention to the proofs of the various properties (additive, etc.) of limits.

Approximately the same amount of space is used to deal with the differential coefficient, general results of differentiation, and differentiation of trigonometric, implicit and inverse functions. Then follows a note on the meaning of a differential coefficient, a single paragraph referring to derivatives of higher orders than the first, one more page on the geometrical interpretation of $f''(x)$ and the question of Maxima and Minima is reached. Here a page suffices to deal with the value of derivatives higher than the second, a footnote pointing out that these considerations are " somewhat outside the Intermediate course in most Indian Universities ". Twenty pages are next filled with a discussion of numbers, Dedekind's theorem being mentioned as " of very great theoretical importance ".

The next chapter (VIII) deals with exponential and logarithmic functions. Chapter IX (pages 183-216) treats Integration as the inverse of differentiation, methods by substitution and parts being explained. Geometrical applications of integration follow, including work on the area under a curve, definite integrals—where the " Fundamental Theorem " of the Integral Calculus is stated as $\frac{d}{dx}\left\{\int_0^x f(t)dt\right\}=f(x)$, approximate integration, and volumes of solids.

Chapter XI reintroduces limits—" in endless or infinite processes ", area under a curve and proceeds via " integrability " to a brief consideration of integration as " the addition of small parts to make up the whole ".

The book concludes with a chapter of Miscellaneous Results, which might better have been incorporated in appropriate places in the main body of the text. For example, partial fractions are here discussed but no mention of their use is to be found in the chapter on integration. Hyperbolic functions, polar coordinates, " surface and volume integrals ", and Taylor's series are all summarised in this final chapter.

There is a large number of examples grouped throughout the book, answers being given at the end of each chapter. No tables are included but there is a fair index.

A disconcertingly long list of addenda and corrigenda is placed at the beginning of the volume and unfortunately even this does not correct all the mistakes, some of which are all too obvious (*e.g.* in page headings). The diagrams (of which there are 53) are often not clear, while the differing types used do not always seem well chosen. The book is printed at the Baptist Mission Press, Calcutta.

The scope of the book was prescribed by the author ; within the range he himself chose, we may say it is a good attempt at a difficult task. F. W. K.

College Algebra. By P. K. Rees and F. W. Sparks. Pp. xi, 312. 12s. 6d. 1939. (McGraw-Hill)

That which is treated in this book is well done. It is hard to visualise a college course in this country requiring just what is contained therein ; for example the initial implication that the idea of a function will be a novelty to the first year student is an immediate sign that the book was designed elsewhere. But judicious selection should enable the book to be used ; it deserves to be.

The function is treated at the beginning of the course in order to whet the newcomer's interest. It is followed by chapters on Systems of Linear Equa-

tions (in which the determinant notion is introduced); Ratio, Proportion and Variation; Exponents, Logarithms; Factors, etc.; Partial Fractions; Quadratic Equations; Higher Degree Equations (this includes the Remainder Theorem, and Horner's method for determination of irrational roots); Simultaneous Quadratic Equations; Progressions, Annuities; Induction; The Binomial Theorem (proved by induction for positive integral indices but without proofs for other "exponents"); Permutations and Combinations; Probability (these last two mentioned chapters including some most interesting examples); Determinants (of order n), and Complex Numbers (a dozen pages only).

Tables of logarithms to base 10, trigonometric functions (for 10 minutes intervals only), Powers, Compound Amount and Present Value of Annuities are included, together with a fairly good index and answers to the even-numbered problems.

There is a good number of problems throughout the book, and many worked examples. These are in general well done, with explanations of the various steps clearly inserted. The various rules, established or assumed, are also clearly pointed out.

Almost all the work may be highly commended, although one may disagree with various treatments.

For example, in the section on "exponents and radicals" a question in the form Evaluate (a) $16^{\frac{3}{4}}$, (b) $8^{\frac{3}{5}}$ is worked in the text to give the answers (a) 8, (b) $2\sqrt[5]{16}$.

Another solution of a problem requiring the "simplified" form of

$$\frac{3x^2y^{-3}z^{\frac{1}{2}}}{6x^{\frac{3}{4}}y^{\frac{1}{4}}z} \quad \text{is given as} \quad \frac{x^{1\frac{1}{4}}y^{\frac{3}{4}}z^{\frac{1}{2}}}{2y^4z}.$$

Neither explanation for the dogmatic rules used (*e.g.* no fractional exponents should occur in the denominator), nor reason for the forms chosen is found.

Again, in the chapter on Determinants, one is sorry to see perpetuated a_{ij} as denoting the general term found in the ith row and jth column. These are not the most convenient letters to use, apart from their other associations.

The book is exceedingly well bound and printed and it should be a pleasure to use it.

F. W. K.

Mathematics applied to Electrical Engineering. By A. G. Warren. Pp. xv, 384. 15s. 1939. (Chapman and Hall)

This is a very interesting book because it deals with the applications of mathematics to such a variety of electrical problems, though it is only right to say that these problems are mainly connected with the telecommunications side of electrical engineering and not the power side, indeed the words motor and dynamo are not in the index.

At the beginning there are four revision chapters on complex numbers and calculus and it is remarkable how many fundamental results have been condensed into these forty pages without any apparent overcrowding; we should have preferred however not to have seen the old-fashioned method of relating a complex number to an exponential function by means of the equation $\log_e k = j$, when complex powers of a number have not been defined. These introductory chapters are followed by an admirable chapter on electromagnetic connections up to Faraday's and Ampère's laws, which we might say is put in to get the reader who is mainly an engineer or physicist into the right frame of mind for reading the rest of the book, namely that of looking at engineering problems from a mathematical standpoint.

In chapters VI and VII the results of the earlier chapters are applied to finding the magnetic forces due to various circuits, calculating inductances and capacities and to solving alternating current problems by means of complex numbers and vector diagrams. These are followed by two chapters on partial differentiation and harder integration, including reduction formulae, gamma functions and double integrals.

Naturally in a book of this kind a large part is devoted to differential equations; there are no less than eight chapters. First, linear equations of the first and second orders are applied to the theory of series circuits, transformers, ballistic galvanometers and valve oscillators; then solution in series and Bessel functions are used in finding the variation of the resistance of a cable with the frequency and in explaining the stress effect at high frequencies, but there is no mention of the variation of the inductance, though this would not necessitate more than an extra page. The partial differential equation of wave motion in one space dimension is solved and used to deduce the plane and spherical wave solutions of Maxwell's equation. Here one finds, as in so many other books, that there is no mention of the impossibility of absolute spherical symmetry in an electromagnetic wave; spherical symmetry is only possible in longitudinal waves such as sound waves; it is not possible with transverse oscillations.

It is perhaps surprising after what we have already said that there are still three long chapters on Fourier series and harmonic analysis, Heaviside's operational calculus and the use of conjugate functions. The middle one of these will probably be read with as great interest as any in the book. The author very wisely develops the use of the operator from the solution of the series circuit with inductance and resistance; this is probably the shortest and most satisfactory method of approach in a book of this kind where one's room is bound to be limited; it suffices to deal with all the problems which can be solved by Heaviside's partial fraction rule. This rule is applied to the telegraph equation, which has already been solved by the more ordinary methods in the chapter on partial differential equations.

Except in one or two minor points mentioned above the presentation of the mathematics in this book will please all mathematicians except the absolute purists, and so far as a mathematician can judge the electrical problems dealt with are chosen with discrimination. There is only one important criticism we have to make: which is that too often the mathematical equations are written down from the electrical statement of the problem as if they were obvious. As any one who has taught mathematics to electrical engineers will know many students will accept this, but the really understanding ones will not; they will probably have been given the equations in this way in their engineering classes and, when they come to their mathematical lecturer, they want to know exactly how the equations are deduced. I can for instance imagine such a student asking why $i=\frac{dq}{dt}$ on one page and $-\frac{dq}{dt}$ two pages further on (actually this is done quite correctly, but there is no explanation); he might also ask which is the positive direction for the currents in the diagram of the transformer equations when the terms in the equations all have the same sign, and what L means exactly in the telegraph equation and how this equation can be related to Maxwell's equations.

In spite of this criticism we do however welcome this very worthy addition to the series of monographs edited by Mr. Young. It should prove extremely useful to students reading for honours in electrical engineering, even if they are not taking mathematics as a subject but still more if they are. The

printing of the formulae and of the diagrams is very clear indeed and makes the book a pleasure to the eye, as well as to the mind. H. V. Lowry.

Theory of equations. By H. W. Turnbull. Pp. xii, 152. 4s. 6d. 1939. University mathematical texts, 6. (Oliver and Boyd)

In this small book, based upon a course of lectures given to a first year honours class, Professor Turnbull has set himself to provide a useful manual for students rather than an exhaustive treatment of the subject. Some selection from the many available topics was clearly necessary, and, in an elementary treatment, invariant and group theory are, naturally, omitted. From other possible themes the author has selected those which he feels have importance both in themselves and as preliminaries to higher algebra. He treats with special care the various types of number, the theories of polynomials and rational functions and the H.C.F. theorems: there is a concise but valuable chapter on symmetric functions, with some mention of the theory of partitions, and some interesting historical notes are given. Other topics are treated on what may fairly be called conventional lines. If the general impression is one of "scrappiness" the fault lies probably in the nature of a subject which touches the fringe of so many other fields of study. There is an outline of Gauss's first proof of the existence theorem and the importance of Liouville's theorem in this context is explained.

The treatment of continuity and the derivative oscillates between something approaching rigour and an appeal to intuition. On the latter basis we are asked to agree (p. 69) that "each real root is a continuous function of the coefficients" though an equally simple diagram would show clearly that a multiple real root may be reduced in multiplicity or lost altogether by small changes in the coefficients. The argument indeed implicitly assumes non-repeated roots and to these a rigorous though elementary treatment can be applied. There is an error of some importance in the proof (p. 18) of the continuity of x^n, where the argument runs as follows:

$$\text{“ } k=(x+h)^n-x^n=h(nx^{n-1}+\ldots+h^{n-1})=hA.\ \ldots$$

... If k is to be numerically less than ϵ we simply choose h numerically less than ϵ/A". Unfortunately A is not known until h is determined.

The general prevalence of one other venial error may pardon its mention. If, in dealing with the amplitude θ of $\{r, \theta\}$, we say (p. 6) "to avoid ambiguity we invariably assume that $-\pi<\theta\leqslant\pi$", we cannot state without qualification (p. 8) "multiply the moduli and add the amplitudes to get the modulus and amplitude of a product".

There are a few misprints of no great importance but the book is well produced and remarkably cheap. We should be grateful for the enterprise which through this series attempts to meet the very real needs of our impecunious students. J. L. B.

Irrationalzahlen. By O. Perron. 2nd edition. Pp. viii, 199. RM. 9.80. 1939. Göschens Lehrbücherei, Bd. 1. (Walter de Gruyter, Berlin)

If the rational numbers are regarded as given, the theory of irrationals may be developed by the method of Dedekind or by a treatment based on convergent rational sequences which derives from Cantor and admits various alternative presentations. In the one case a real number is a section of rational numbers, in the other a class of equivalent rational sequences. When the relations of magnitude and the laws of operation are properly defined the resulting arithmetics are identical in form. The third alternative of a purely axiomatic development of arithmetic is of course open but, if this be adopted, the older theories retain their value since they exhibit systems which, by

satisfying all the axioms, guarantee the freedom of the latter from contradiction.

This book develops in great detail the theory of sections with the corresponding theory of bounds, limits, powers and logarithms. A natural sequence of ideas leads to a discussion of the expressions of numbers as decimals and continued fractions as well as in the less familiar forms of Cantor's series and product and the series of Sylvester, Lurōth and Engel. The criterion for irrationality is given in each case.

An irrational number may be either algebraic or transcendent; since algebraic numbers are enumerable, transcendent numbers must exist. The final chapter discusses their construction with special reference to Liouville's numbers and the criteria for them. The work of Minkowski and others on criteria for algebraic numbers is mentioned but not discussed and the chapter concludes with proofs of the transcendence of e and π.

The longest section of the book deals with the problem of approximating to irrationals by rationals. Every irrational number ξ can be approximated to in an infinite number of ways by a rational fraction p/q, the error being less than q^{-2}. Further, if η is any number and ϵ is arbitrarily small, there are integers x, y such that $|\xi x+y-\eta|<\epsilon$. Each of these theorems admits of extension, in modified form, in the one case to a set of irrationals (ξ_r) and in the other to a set of forms with matrix (ξ_{rs}). Discussion of the second case involves consideration of the rational rank of the matrix. The results given in the first edition were all classical and culminated in the theorem that, to use the language of n dimensions, if $(1, \xi_1, \ldots, \xi_n)$ are rationally independent, the points with coordinates

$$(\xi_1 x_{n+1}-x_1, \ldots, \xi_n x_{n+1}-x_n)$$

are everywhere dense in space. In this edition a proof is given of Weyl's theorem that, speaking crudely, the points are everywhere equally dense. Apart from a substantial increase in the bibliography, this section is the only addition of importance and those who possess the first edition may rest well content therewith. To others the book may be heartily recommended, not only as a handy work of reference where we may find how to do the things we know can be done but are too lazy to do, but as a useful introduction to interesting and important problems. J. L. B.

Mathematics for Technical Students. II. By A. Geary, H. V. Lowry and H. A. Hayden. Pp. viii, 417. With tables. 5s. 1939. (Longmans)

This volume is intended to cover the second year's work of a three years' national certificate course. Part I was reviewed in the *Math. Gazette*, Vol. XXIII, p. 112; and the present volume maintains the excellent impression produced by the preceding one. It intentionally includes more than would normally be attempted in one year in any particular institution, in order to allow for variations in the syllabuses of different institutions.

It starts with a revision chapter on Algebra, and then proceeds to deal with factors and fractions; the solution of quadratic equations, with a few examples of solution (by factorisation or by graphical methods) of equations of higher degree; logarithms, with some harder examples than those of part I and including work on changing the base of a logarithm, with special reference to base e; variation, and the determination of laws to fit experimental data.

The section on geometry covers the chief properties of similar triangles and similar figures; some miscellaneous theorems on chords and tangents of circles and the centroid of a triangle; and concludes with an elementary chapter on solid geometry.

The longest section of the book is devoted to trigonometry. This can be read without previous knowledge of the subject, so that for those who have already worked through the trigonometry of part I the earlier portion will serve as a revision course. The section covers the trigonometrical ratios for angles of any magnitude and their graphs, equations, vectors, the trigonometrical ratios of the sum or difference of two angles, the sine and cosine rules and their application to the solution of triangles, and formulae for the area of a triangle.

The last two chapters of the book are on differentiation and integration respectively. The approach is graphical, but the rules for differentiation and integration of ax^n are stated, so that some simple examples of the applications of differentiation and integration can be given.

As in the first volume, the examples are numerous and well selected, and designed to maintain the interest of the student by showing something of the types of application which can be made of the work he has been doing. A set of four-figure tables is included at the end of the book.

Teachers in technical institutions who have not already made the acquaintance of this work would be well advised to do so now. W. H.

Elementary Statics: a Text-Book for Engineers. By M. Appleby. Pp. viii, 164. 7s. 6d. 1939. (Cambridge)

The scope of this book is the Intermediate syllabus, or that of the Qualifying Examination for the Mechanical Sciences Tripos, and it is evidently designed for those who begin the study of mechanics late, since trigonometry is freely used. The author in his preface disavows any attempt to " develop the subject from the minimum number of fundamentals "—which surely demands no apology in a book intended as an introduction to the subject! So in the chapter on " Translation and Rotation ", the principles of the force polygon and of moments are developed separately; that is all right—but it is a pity that their essential equivalence is not, somewhere in the book, explicitly pointed out; the approach and the illustrative material are, however, quite good. In the next chapter, on " Work and Machines ", the screw and gearing receive more attention than is often the case. Then follows a chapter on " Friction ", in which the force of friction when *not* limiting, is first emphasised, and in which friction in journal bearings (but treated as *solid* friction) is treated in some detail. " Centres of Gravity " comes next, followed by " Graphical Statics ", the latter dealing with the method of sections as well as Bow's notation and the link and vector polygon. The chapter on " Couples and Pure Rotation " includes an introduction to bending moment and shearing force in beams, and the final chapter, headed " Miscellaneous ", contains brief notes on, among other things, Hooke's law, the suspension bridge, virtual work, and the energy test of stability. Each chapter contains a large number of worked examples (which are frequently actual examination questions), and is followed by a generous batch of exercises to be worked by the student; very many of the exercises have a real practical flavour, and this is also true of the miscellaneous exercises at the end of the book; answers are given.

The book would seem well suited to the needs of the type of student for whom it is written, *i.e.* the embryo engineer, or indeed for any student commencing the study of mechanics after taking school certificate. As a textbook for class use in the pre-certificate stage, it is hardly suitable, but any teacher of elementary statics will probably find much in the book to assist him. Many of the interesting exercises based on real problems could be worked graphically if trigonometrical knowledge is lacking.

But, just because the book has so many good points, it is a pity that a little

more care was not expended upon its writing. The English is often unpolished—and even faulty. Inconsistency is not rare, such as (p. 117) defining the "moment of a couple" (this in Clarendon type), and then using the word "torque" (undefined) for the rest of the page; or again, the uncertainty as to whether a system of co-planar forces which reduce to a couple has "no resultant" or a "resultant couple", which is created by the author's use sometimes of one, and sometimes of the other, description. Opportunities have been missed; one such has already been mentioned. The connection between the link or funicular polygon, and a linkwork or string, is not mentioned (a connection which puts a physical background behind the proof!), nor is the connection between the link polygon and the bending moment diagram mentioned (although this is the ordinary engineer's method for actual drawing!).

W. G. B.

Tables for converting Rectangular to Polar Co-ordinates. By J. C. P. Miller. Pp. 16. 2s. 1939. (Scientific Computing Service Ltd.)

The necessity for converting rectangular to polar coordinates arises in many computational problems, and can be either a source of irritation if it arises only occasionally or a major item, to which special consideration must be given, if it occurs in bulk. The present tables have been designed by a practical computer, who has had vast experience of this work, and it can be said that, within their limitations of 4-figure accuracy, they represent the best that tabulation can provide for the solution of the problem, either for occasional or routine use.

The method employed is, of course, that of forming $k = \tan \theta$ by division of the smaller of x or y by the greater, and then using the tables to give θ and $\sqrt{(1+k^2)}$, from which r is formed by multiplication by the greater of x or y. In the method there can be no original merit, which lies entirely in the arrangement and presentation of the tables. Four points, in particular, enhance the value of the tabulation: the small interval of 0·001 in k, resulting in the differences never exceeding 10; the simple and easily followed method of indicating the correct "octant" (due to the four possible combinations of signs of x and y and their relative magnitudes); the tabulation (in degrees and decimals) of ϕ, the complement of θ, to avoid subtraction; and the provision of an alternative column giving θ in radians.

The tables are excellently printed, they are prefaced by a short introduction which gives all the explanation and examples necessary, and they can be recommended to all who have (or who may have) computation to do. The only adverse criticism is to their limitation to four figures; a fifth figure could have been catered for without extension, which would have (in the case of the reviewer) given just that little extra!

A word of praise is due to the publishers, who have made publication of small, useful tables like these possible.

D. H. S.

Modern Machine Calculation. Fast and simple methods for every day problems with the Facit LX. By H. Sabielny; translated and revised by L. J. Comrie and H. O. Hartley. Pp. 74. 5s. 1939. (Scientific Computing Service Ltd.)

This small booklet is in four sections; the first of these is devoted to a detailed description of the latest model of the Facit calculating machine, the second contains a fairly comprehensive selection of examples of practical computation, while the third gives an interesting account of the methods and "tricks" that have to be adopted to utilise a machine efficiently in dealing with the British systems of currency, weights and measures; the last section

contains the necessary conversion tables for the decimalisation of the odd units that occur, as well as several interest tables.

The Facit is a hand calculating machine with unusual features—the principal one being a ten-key setting as contrasted with the setting levers usually found on barrel-type machines, and with the full keyboard on most electric machines. The first section describes in detail the operation and care of the machine, and can confidently be recommended to all to whom the Facit is not familiar. Opinions as to the merit of the ten-key setting may legitimately vary, but there is no doubt that the Facit is a machine in which the utmost is made of the advantage accruing from this feature ; before choosing a hand calculating machine, this machine must be given careful consideration, and this section will provide an excellent introduction.

The remaining sections are of more general interest, and, with one or two exceptions, contain nothing that cannot equally well have been written about any machine. It is this combination of a specialised description of one particular machine, together with general examples of machine computation and tables that are independent of the particular machine, which makes it difficult to place the booklet in any definite category. As an instruction book for the Facit machine one could hardly improve upon it, but such booklets are not usually sold at a relatively high price ; yet it is far more than one could expect to be given away with the machine. The first section has a general appeal to both new and experienced machine users, and would form an admirable advertisement folder for the particular machine ; although the second section is written so as to be directly applicable to the Facit, the details of setting and position are unnecessary to even an inexperienced user who has read section one with the machine in front of him, and (with a few exceptions) it is consequently reduced in value for the general reader. The booklet is one that can be appreciated to its full only by new users of the Facit (who would expect something of the kind with the machine at a much cheaper price); one section only is of interest to experienced computers, and three sections (of which the larger is unduly lengthened and complicated by being made applicable to one machine) of permanent interest to the practical computer. The extension of what is primarily an instruction book to an approach to a textbook thus appears to have increased its cost, without greatly widening its appeal.

Of the presentation of the matter, of the choice and set-out of the tables, of the printing and of the careful explanation of the examples there can be no criticism. In fairness to the translators, who have in some ways extended the original, it must be said that they must have found the limitations of scope exceedingly irksome. D. H. S.

Astronomy. By R. H. Baker. 3rd edition. Pp. x, 527. 16s. 1939. (Macmillan)

As a descriptive introduction to astronomy, we find this a most attractive volume. Starting from the aspects of the sky and the apparent movements of the sun, planets and stars, the author gives a useful summary of astronomical knowledge reaching out to the great stellar systems which people remote space. The numerous illustrations deserve a special word of commendation. The book is intended primarily for American students as an introductory course in astronomy. The astrophysical aspect is given very fully ; there is a complete absence of simple problems requiring spherical astronomy. While it is a matter for debate as to the best approach to a study of astronomy, it is certain that the student will be attracted by the astrophysical aspect,

though he will be the loser in being shown, for instance, the transit instrument shorn of the geometry of its three errors of collimation, level and azimuth.

The revision required for this 3rd edition has in general been carefully done. There is a photograph of reflexion nebulae taken with a Schmidt camera—the important new accessory to the telescope. As other instances, H. H. Plaskett's observational work on the solar granulation, and Lyot's distinctive work on the solar corona also receive mention. But the anachronism of fixed wires in the field of a transit instrument in textual proximity to a quartz crystal clock cannot be overlooked! H. N.

Higher Certificate and Intermediate Tests in Mathematics. By R. J. Fulford. Pp. 90. 1s. 6d., with answers. 1938. (University Tutorial Press)

This is a cheap, well bound and clearly printed collection. There are groups of eight papers of about $1\frac{1}{2}$ hours' length on each of Algebra, Trigonometry, Analytical Geometry, and Calculus, with five papers each on Plane Geometry and Solid Geometry. These are followed by ten mixed papers and ten more difficult papers of about 3 hours' length. The easier papers could be tackled by second-year sixth-formers, but the harder ones are well below the higher certificate distinction standard. The papers in each subject are not arranged in order of difficulty, and the questions in each paper are not related to one another. A. P. R.

A Mathematical Handbook for Sixth Forms. By E. W. Burn. Pp. 164. 2s. 3d. 1939. (Harrap)

The nature of this book and the ideals of its author are best indicated by quotation from the preface. " Part I contains results systematically arranged, with notes and proofs; these often explain easy things which are neglected in text-books, but cannot be neglected in practical teaching." " The sixth-form boy who is working for an examination must learn to arrange his information well, and he must know rather more than the examiners require." " Part II gives concise proofs of the theorems asked for by examiners. The proofs are written out exactly as they should be written out in an examination." The book bears evidence of hurried compilation, there are many loose and misleading statements, and the style is abrupt, even dictatorial. The arrangement and printing are not agreeable to the eye, and the numbering of results and paragraphs is often confusing. The scope is indicated by the absence of Determinants or of Taylor's Theorem, and Mechanics is not included. The book is in no sense a miniature " Workman ". A. P. R.

An introduction to the Theory of Functions of a Real Variable. By S. Verblunsky. Pp. xi, 169. 12s. 6d. 1939. (Clarendon Press)

This is a very good book and one which fills a long-felt need. It is scarcely necessary to say that it is a book with a conscience. The author is meticulous in all his proofs and statements and it is a pleasure to read a comparatively elementary account of the subject feeling that the author knows better than his reviewer. But there is, in addition, real artistry in designing the structure so that each theorem comes where it pulls its weight with greatest effect. The crisp style used might well be copied by other authors.

Certain points are unusual and specially to be praised for neatness. The exponential and trigonometric functions are defined by series but for their own sakes long before the general and more difficult theorems on series are considered. The proof that $\exp(a+b)=\exp a \cdot \exp b$ is obtained by differentiation. A similar process gives the addition laws for the sine and cosine.

The proof of the existence of a smallest positive x, such that $\sin x = 0$ is very neat. The reviewer welcomes the manner of approaching the definite integral separating the existence aspect and the particular method of calculation.

In one respect the title of the book may mislead those who expect such things as derived sets of points, or non-differentiable continuous functions and so forth. The author has wisely confined himself to the main lines of development and a student educated from this book could easily learn to handle several dimensions, etc.

There is one point which would require considerable amplification by the teacher and that is the postulate of the continuum on page 10. It is clever and avoids discussion of Dedekind cuts but only an experienced mathematician would see the sense.

The book begins with two chapters on number, chiefly inequalities, and on sets and functions with the emphasis on the latter. There follow accounts of continuity, the derivative and the primitive. The last two chapters are on integrals and series.

The book is particularly suitable for the young mathematician at the University. For those at school it is too advanced, taken as a whole, except for the few. But school teachers could profitably pick out parts of it for presentation to replace proofs now fashionable. What is more important, these teachers could be inspired by the book to treat more elementary questions in a similar though necessarily modified way. The book would not only give aesthetic pleasure but even some instruction to University teachers. P. J. D.

Integration. By R. P. Gillespie. Pp. xiii, 126. 4s. 6d. 1939. University Mathematical Texts 3. (Oliver and Boyd)

After a general introduction the book considers integrals, including multiple, curvilinear and surface integrals from a non-rigorous point of view. Then in Chapter V the Riemann integral is defined and discussed at considerable length. There then follows a chapter on infinite integrals including Eulerian integrals and a more miscellaneous chapter with double integrals, uniform convergence, differentiation under the sign of integration and orthogonal functions, with other topics.

It may be said at once that the author has attempted an impossible task. In so small a compass it would have been sufficiently difficult to present the subjects without the more difficult proofs, these being left for reference to larger treatises. But the author has valiantly tried to give rigorous proofs as well. To some extent he has succeeded but it is clear that the space allowed has seriously cramped his style. He often omits important conditions, assuming them to be understood, and quite considerable and difficult arguments have had to be so curtailed that the student would require extra assistance.

From the less rigorous point of view the discussion on pages 4 and 5 is not clear and the distinction between the definite and indefinite integral is not explicitly pointed out, though it is vaguely implied. On page 10 in the table of standard integrals not only is nothing said about the arbitrary constant but one integral is said to be "$\sin^{-1}x/a$ or $-\cos^{-1}x/a$" without any explanation of the mysterious "or".

Some of the examples are tricky and some on page 25 require the reasoning of later chapters to make them intelligible.

On the rigorous side there are frequent omissions such as the case in which an integral oscillates infinitely or the statement which should have been made on page 97, that $f(x)$, $g(x)$ are to be bounded. Here the author gives a sketchy

"proof" that if f and g are integrable so also is fg. However all that is proved is that if f and g oscillate boundedly so also does fg. The given inequality might have led to a proof with an expanded argument and a different definition of Riemann integrability. This is the sort of thing so small a book cannot handle suitably. It would have been better to use only continuous functions.

On pages 104, 105 the proof concerning Fourier series is too sketchy for any reasonably critical student.

Nevertheless the reviewer would like to praise the courage of the author in leading a "forlorn hope". Moreover there are places here and there where the brevity helps to show up the essential points. Certainly the book gives very much for a very small price. P. J. D.

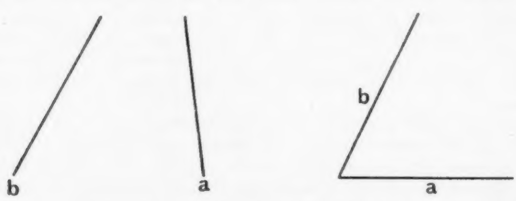
Elementary Geometry. By H. A. Baxter. Pp. vii, 215. 4s. 1939. (Blackie)

This rather unusual textbook in geometry covers a very wide range of syllabus in its 215 pages. In addition to dealing with the whole of the course necessary for the School Certificate examination, it includes many of the standard properties of solid geometry usually considered necessary for the Higher School Certificate examination. Taken as a whole, it gives the appearance of attempting to cover too much ground in too small a space but closer consideration shows the book to be very compact and concise.

One original feature is the sequence. It is only necessary to mention a group of successive headings from the book to illustrate the unusual nature of the development: Equal Chords, Similar Triangles, Areas of Similar Triangles, Inequalities, Angles in Segments, Mean Proportional, Concyclic Points. An early introduction to ratios is included.

For beginners, the book needs considerable supplementation by the teacher. Fuller treatment in the text of the more elementary constructions and properties would have been welcomed. Bearings in problems on scale drawing (*e.g.* N. 23° W.) are introduced without any reference to the meaning of such bearings. For the more advanced pupils, the examples are, on the whole, quite good. There are numerous exercises to each group of theorems and they vary from easy straightforward examples to more difficult ones, the range being well graded.

The definition of an angle may be misleading to some pupils. An angle is defined as follows: "The amount of turning which a straight line has to do in order to move from the straight line a to the straight line b is called the



Angle between the lines a and b". Such a definition is apt to cause confusion, especially as the angle in the first figure might be α, $2\pi-\alpha$, or $\pi-\alpha$. Parallels are defined on the basis of the angle between the lines being zero. The dangers implicit in such a definition are well known. In view of the fact that they have been discussed in previous issues of the *Gazette*, there is no necessity to enlarge on them. The reader need only refer to some interesting notes in the *Gazette*,

XII, No. 172, Note 727, by Professor D. M. Y. Sommerville and Mr. W. J. Dobbs, dealing with precisely the same definitions suggested then by the author of this book.

There is an error in sign in the proof of Theorem A (p. 176). It should read: " For any consideration tending to show that $\angle ROM > \angle RON$ ... ". H. B.

A Stage " A " Geometry. By L. R. SPENSLEY and E. N. LAWRENCE. Pp. vi, 104. 2s. 1940. (Macmillan)

A small but very presentable and clearly illustrated book which covers all the needs of pupils requiring their first introduction to Geometry. It should be highly suitable for the first year of a normal five year course to School Certificate. Very little deductive work is attempted, but the treatment is such that any pupil following it should have a sound foundation on which to build his more formal geometry.

Perhaps the most interesting feature of the book is the excellent and original treatment of solids. A whole chapter is devoted to the development and making of solids from plane sheets of material and the general treatment is particularly attractive. Questions which are carefully planned and interesting to the beginner in solid geometry are such that he is made to see the various solid figures from all angles, an achievement which most teachers will welcome. Such a treatment of solids in the first year should be a valuable asset to any pupil who has later to deal with the more complex questions in three-dimensional geometry.

No special attempt has been made to deal with field work in detail but its allied branches of scale drawing, bearings and angles of elevation and depression receive full attention. H. B.

Advanced Algebra. By S. BARNARD and J. M. CHILD. Pp. x, 280. 16s. 1939. (Macmillan)

This is a continuation of the *Higher Algebra* by the same authors. It bears the same marks of sound knowledge, careful presentation, and good workmanship. The range of subjects covered is wide. From the mass of topics that offer themselves to the writers of an advanced algebra for mathematical specialists at school and university the authors have made a reasonable and attractive choice. It could not be expected that the choice would commend itself as ideal to everyone.

Perhaps the best service that a reviewer can render his readers is to note what is included and what excluded. There is nothing about matrices: the authors, rightly I think, hold that these are best studied in texts devoted specifically to matrix theory: there is nothing about determinants, as such (beyond what has already been done in the *Higher Algebra*), save what enters under the heading of elimination, invariants, covariants and canonical forms. There are chapters on bilinear substitutions, complex variable, systems of quadratics (mostly in one variable, but some of them homogeneous in two), double series and uniform convergence (a lone voice crying in a hostile wilderness, I deplore the admixture of algebra and convergence in advanced textbooks), exponentials and logarithms, elimination, probability, continued fractions, quadratic residues, indeterminate equations, primitive roots of equations, Gauss' treatment of $x^{2n+1}-1=0$ and, in particular $x^{17}-1=0$ and $x^{19}-1=0$, sums of squares, invariants and covariants of binary forms, and, finally, a chapter on homographic transformations from a geometrical rather than an algebraical standpoint.

As the reader will see, this is a very mixed bag, and few undergraduate supervisors will feel like recommending their charges to a steady cover-to-cover

reading of the book. But it is unlikely that the authors ever intended that they should.

If a reviewer may permit himself a criticism of a very commendable book, it is that the authors have tried to do too much, to broach too many topics, to open up too many interests in the space of one volume. It has often occurred to me in recent years that the hardest problem of the mathematical teacher and examiner is to prune the mathematical tree so that the new wood is not stifled while yet the best of the old wood is preserved. The trouble is that no two pruners will agree on where the cut of the pruning knife shall fall.
W. L. F.

Vector Analysis. By J. H. Taylor. Pp. ix, 180. $2.85. 1939. (Prentice-Hall, New York)

This is an introduction to Vector Analysis formulated with a view to leading naturally to its extension Tensor Analysis. The subject is approached in the axiomatic manner with refreshing emphasis on the importance of the notion of linear independence, and the idea of a basis which flows from it.

The author defines a vector as a quantity whose instances (states) admit of a [1, 1] reciprocal continuous correspondence with a set of translations in 3-space. The laws that shall govern addition and multiplication with scalars and their relation to points in a given affine coordinate system are then postulated. The introduction of the fundamental quadratic form accompanied by definitions of length and angle leads to the Cauchy-Schwarz inequality and the invariance of the scalar product of two vectors under the group of affine transformations.

Care has been taken not to make use of a specialised choice of base vectors or coordinate system until it is clear that such a choice will result in simplified description or procedure. In this way the invariantive character of the theorems and methods receive an emphasis which is frequently obscured by the exclusive use of the **i**, **j**, **k** system of base vectors.

The book consists of four chapters: (I) the algebra of vectors; (II) differential calculus of vectors, with applications to curves and surfaces based on the work of Blaschke; (III) integral calculus of vectors, treated in an intuitive manner, with an account of the operator "nabla"; (IV) an introduction to tensor analysis.

The treatment is clear and elementary, and presumes only that a first course in calculus has been studied. There are copious illustrations and exercises for the student. These exercises are well arranged and amplify the text quite considerably.
L. M. M-T.

Mécanique statistique quantique. By F. Perrin. Pp. 224. 100 fr. 1939. Traité du calcul des probabilités, tome II, fasc. 5. (Gauthier-Villars)

In this book Professor Perrin gives, with typical Gallic clarity, an exposition of quantum statistics and a number of its more interesting applications. He spends the first five chapters summarising the technique of statistical mechanics from a viewpoint essentially that of Gibbs. Particular emphasis is laid on the theory of perfect gases and on the theory of radiation in view of their relation to the quantum statistics discussed later. After a brief chapter summarising the principles of quantum theory the appropriate modifications of the statistical formulae are given in Chapter VII. A particularly satisfying feature is the way in which the Bose-Einstein statistics for matter is related to the quantum statistical theory of radiation. Thus the description of the latter theory in Chapters VIII and IX leads on to the new statistics, discussed

in Chapter XII. The principles involved in both the Bose-Einstein and Fermi-Dirac modifications become then very clear and almost natural. The remainder of the book, involving four further chapters, is concerned with applications of the new statistics. In view of the topical interest in the peculiar properties of liquid helium and its supposed origin in the degeneration of condensed helium as in a Bose-Einstein gas the discussion of this case given in Chapter XIV should be of much value. A clear exposition of the application of the Fermi-Dirac statistics to the determination of atomic fields and the density distribution in a neutron star is also given. Finally in Chapter XVII a concluding discussion on the significance of the indistinguishability of particles in quantum theory provides a fitting conclusion to a book largely devoted to the consequences of this explicit recognition in statistical theory of the equivalence of similar particles.

The book can be recommended to both mathematicians and physicists for, while the mathematical treatment is rigorous and logical, it is never allowed to obscure the physical principles involved. H. S. W. M.

Cosmic rays. By R. A. MILLIKAN. Pp. viii, 130. 8s. 6d. 1939. (Cambridge)

Professor Millikan's little book, which is based on lectures delivered during 1936–1937 in America and in Dublin, makes very pleasant reading for its style is light and unorthodox. Interspersed among descriptions of scientific results, one finds musings on political and sociological problems pronounced with an air of authority which is refreshing, even if it is difficult always to agree with the writer.

Although a number of conclusions given in the book now need revision owing to the rapid growth of our knowledge of cosmic rays, the main body of information, described so clearly and simply, still stands. Probably no other branch of modern research in physical science has such a romantic appeal as the study of the rays which boast the singularly appropriate and non-committal name " cosmic ", and Professor Millikan is at pains to bring out this aspect of the subject.

Even though no mathematics appears in this book there is little doubt that mathematicians of every kind would receive entertainment and edification from its perusal and study. H. S. W. M.

Statistical Mathematics. By A. C. AITKEN. Pp. vii, 153. 4s. 6d. 1939. University Mathematical Texts, 2. (Oliver and Boyd)

In reviewing Plummer's *Probability and Frequency*, Professor Piaggio (*Gazette*, 1940, xxiv, p. 63) considers it to be the duty of a reviewer to draw particular attention to any book which gives even a part of what is required in the way of a mathematical treatise on statistics. Dr. Aitken, in one of the first volumes of *University Mathematical Texts*, under the general editorship of himself and Dr. D. E. Rutherford, has given a great deal of what is required, although in condensed form, and the result is a very interesting and readable short book. It should be noted that the title is " Statistical Mathematics ", not " Mathematical Statistics ", and the distinction should be borne in mind, though it may be a subtle one, as anyone can see by comparing the subject-matter with that of any ordinary textbook on Statistics. In his contributions to this subject Dr. Aitken has preferred to regard himself as a mathematician rather than an applied statistician, and the aim of the book is to describe and develop those aspects of mathematics which are directly applicable to statistical theory and practice. The book should be in the hands of every serious

student of statistics, who may then proceed to fill in the details of what is an excellent mathematical framework by reference to other sources.

A useful first chapter develops the axioms of probability, and the only thing here that the reviewer would take exception to is the use of the word (or words) *à priorist*. The use of generating functions of probability, with their applications to both moment and semi-invariant generating functions, will introduce a wider public than hitherto to what is perhaps the most elegant part of statistical mathematics, and to a powerful method of deriving sample distributions of the common estimates of population parameters. The author prefers the term seminvariant, but here again there are objections; R. A. Fisher uses the useful word "cumulant". The book develops in further chapters in the orthodox way of dealing with frequency distributions, calculations of moments, special probability distributions and practical curve fitting. In spite of the title, the practical needs of the statistician in computational problems are kept fully in mind. The author goes on to problems in two and more variates, and deals with correlation and regression, including the method of least squares and polynomial fitting. A final chapter deals with the important subject of the probability distributions of statistical coefficients, deriving the main practical results and tests, and even finding space for the application of the method of analysis of variance to agricultural experimentation.

It seems ungracious in a reviewer to ask for more when so much is given; in any case it is far easier to criticise such a work as the present one than to write one. No one will doubt the value, at the present stage of development of the theory of statistics, of such a handy compendium of established theory and of practical method. But is it too much to hope for that the author will one of these days present the world with an expanded version? The book is in some places a little discursive, while in others it is all too brief. A few minor points of criticism may be made. There are typographical errors in the example on p. 55; it would assist the reader to give one example of the calculation of z on p. 74; the word "a" is missing in the last line of p. 78; the discussion of the error of sampling of r on p. 92 is out of place in the sense that the theory behind it is postponed to Chapter VII, and the reader meets the treatment of r before he has come to even the simplest case of the distribution of the mean; the description of the data on pp. 99-100 is not very clear (a contingency table usually consists of frequencies), and columns and rows appear to have got interchanged in the text on p. 100; the fact that the table on p. 99 consists of probabilities, and not frequencies, should be mentioned on p. 104 in connection with the calculation of χ^2; the word "of" is missed out on p. 111 (line 7)—in this section the author is too modest to give his own elegant methods for the solution of normal equations; the coefficient table on p. 118 is out of alignment with the numbers bordering it; on pp. 128-9 the use of a binomial example to illustrate the sampling distribution of means from normal populations might be improved on (this applies also on pp. 133-4); and on p. 143 it should be mentioned that the second transformation of r only applies when $\rho=0$. A final word of criticism may perhaps be permitted on the treatment of Analysis of Variance. In a book called *Statistical Mathematics*, it would be appropriate that this method be introduced generally as an extension of the comparison of two means to the comparison of any number of means by means of a single comprehensive test of homogeneity. It is true that it was invented by R. A. Fisher for use with experimental designs in agricultural field experimentation, but it might help others to see its general applicability if it could take its place with other tests of significance in the way indicated, and indeed be further exemplified by showing that many tests of

significance can be cast into the form of an analysis of variance. The description of the method given in the book under review is useful, but all too brief.

J. WISHART.

Tables of Random Sampling Numbers. By M. G. KENDALL and B. BABINGTON SMITH. Pp. x, 60. 3s. 9d. Tracts for Computers No. xxiv. (Department of Statistics, University College, London)

The science of statistics has been materially advanced at times by experimental rather than mathematical work. Large numbers of random samples are " drawn " artificially from some experimental " population ", deliberately constructed so as to be of the character required, and comparisons are made between observed frequencies and expectation. So much of this sort of thing was done in Karl Pearson's day that a list of " random numbers " was prepared by Tippett for the purpose, and was later published as No. xv in the " Tracts for Computers " series. Others besides mathematical statisticians have found this list helpful, and in particular the development of experimental design in biology (agricultural field experimentation is an example), in a way which necessitated a random set-up of an experiment, has led to Tippett's Tables becoming very well known and widely used. Tippett's series consisted of 40,000 numbers. While for many purposes this is of satisfactory length, and indeed a shorter series is all that is required by the biological worker (Fisher and Yates, in *Statistical Tables*, have provided a new table containing 15,000 random single figure numbers), there have been lengthy sampling investigations for which Tippett's series has not been found adequate. The authors of Tract No. xxiv have therefore provided a new series of 100,000 random digits, and have furnished a guarantee of the random arrangement; four searching tests of local randomness have been applied. The numbers have been arranged in thousands, so spaced and printed that they can be used as single, two-figure or four-figure numbers, and the result is a series that is likely to be so extensive that all ordinary, and most extraordinary, requirements are met by its issue. An introduction describes the method of construction and the tests applied for randomness. Printed by the Cambridge University Press, who have a deservedly high reputation for the production of numerical tables, the publication satisfies the most exacting standards, and is a very pleasing example of typography.

J. WISHART.

Principles of the Mathematical Theory of Correlation. By A. A. TSCHUPROW; translated by M. KANTOROWITSCH. Pp. x, 194. 12s. 6d. 1939. (Wm. Hodge and Co.)

This book, by the distinguished Russian mathematical statistician, created quite a stir when it was first published in German in 1925, and Professor Karl Pearson in particular considered it to be an important work. Denied to many who were unable to read German the book has now, since the death of the author, been translated into English by Dr. Kantorowitsch, who has substituted a useful survey of contemporary English literature on the subject for the author's introductory notes, except that bibliographical details relating to separate chapters have been translated from the original. These notes are collected at the end, following a series of appendices on mathematical points referred to in the main text.

The development of the theory of statistics has been a rapid one, and the fourteen years between 1925 and 1939 have seen many advances, but that does not mean that the present work is out of date. On the contrary, there are few authors who have attempted, as Professor Tschuprow did, to reason closely on the fundamental nature of the correlation concept. The tendency

too often is to make a rather uncritical application of the standard methods of calculation to the data of the experimenter, and to make deductions on the basis of the results of computation without discussing their validity. It can hardly be emphasised too strongly that a careful study of Tschuprow's arguments will turn a moderately good statistician into a much better one. The book is therefore to be recommended to all practical workers, but in addition to this it should be studied by the statistical mathematician as a fundamental contribution to the theory of correlation.

The translation has been well and carefully done, and there are few blemishes, most of these being typographical. J. WISHART.

The Factorial Analysis of Human Ability. By GODFREY H. THOMSON. Pp. xv, 326. 16s. 1939. (University of London Press)

For a considerable number of years there has been a mathematical school in psychology (possibly more than one school) which has considered the problem of the study of human ability along the lines of constructing a factor theory. Professor Spearman has been the principal exponent of a theory of two factors, which postulates that there is a general factor underlying all ability, to which specific factors are added. Professor Thomson was critical of this theory in that he asserted that group factors must be present, a fact which is now generally recognised by exponents of the factor theory. The theory has undergone considerable development, both mathematically and experimentally, and in particular statistical developments have taken place to deal with the considerable volume of experimental data which has been collected to test the theory. The result has been a considerable number of papers and books on the subject. Professor Thomson has now taken advantage of a year devoted to study and research to discuss the whole subject in such a way that it can be studied with profit by non-mathematicians. For the professional mathematician an appendix is provided, though the hope is expressed that he will read the text as well.

The impulse towards the study of factorial analysis comes, at any rate in Professor Thomson's case, from the practical desire to improve the selection of children for higher education. A book like the one under review should therefore be of interest to all educationalists, and in particular mathematical teachers will be interested in a branch of psychology in which so much use is made of their own field of study. The book is interestingly and authoritatively written throughout, and is a pleasure to read. Also it is well documented with references, and dealing as it does with a branch of study which the author has made so much his own, it forms an invaluable source and reference book to all who would like to master this particular field of study. The problem of factorial analysis may be briefly described by means of quotations from the author's contributions to last year's symposium to the British Psychological Society. The scores of n tests for a person may be regarded as providing co-ordinates of a point in a space of n dimensions, and a population of persons will be represented by a scatter of such points. The problem is then to choose a set of axes or factors, preferably orthogonal, to replace the tests as definers of the space, that is of the qualities of the person. Certain principles are adopted in order to select one of these sets as the best. These are : (i) representing the experimental facts (within the limits of error) by a smaller number of factors than tests, (ii) reproducing as much as possible of the whole variance with each successive factor, (iii) reproducing the correlations with the minimum number of common factors, (iv) insisting on a general factor, (v) rotating the factors until they become psychologically significant, (vi) requiring "simple structure", *i.e.* certain mathematical relations between factors and

tests, (vii) requiring invariance of analysis of the same tests in different batteries, when used on equivalent samples of persons, and (viii) using factors and loadings which are reciprocal for persons and tests. Different authorities have concentrated on the application of one or another of these principles, and not the least of Professor Thomson's achievements is the way in which he has been able to bring together their various contributions to the subject.

There are five parts to the book. The first, on the Analysis of Tests, deals with the theory of two factors, its extension to multiple-factor analysis, the sampling theory, the geometrical picture and Hotelling's "principal components". The second covers the Estimation of Factors. The third, on the influence of sampling and the selection of the persons, covers among other things the work done on the sampling error in the theory of two factors, and goes on to deal with the influence of univariate and multivariate selection. A fourth part is concerned with correlations between persons, while an important last part is on the question of the interpretation of factors; the final chapter, appropriately entitled "Stop-Press", summarises recent contributions on the general subject up to the date of going to press. J. WISHART.

Statistical Testing of Business-Cycle Theories. 1. A Method and its application to Investment Activity. By J. TINBERGEN. Pp. 164. 3s. 6d. 1939. League of Nations Economic Intelligence Service, Geneva. (Allen and Unwin)

This publication is the first of a projected series of pamphlets arising out of the examination of existing theories concerning the nature of the trade cycle in a book, *Prosperity and Depression*, by Professor G. von Haberler. Following on the publication of these theories it was considered that the next stage was to confront them with the historical facts—"to subject them, in so far as those facts can be quantitatively expressed, to statistical analysis". The work under review explains the statistical method it is proposed to employ, and its interest to mathematicians lies, not so much in the application of this method to the economic problems considered by the author, as in the method itself. The author explains the part which the statistician can play in business cycle research, and shows that he can only provide "verification" of a theory in a limited sense. But beyond this he can search for causal relations between one dependent series and several causes. It is desired to discover, not only what causes are operative, but also with what strength each of them operates. This leads to the method of multiple correlation analysis, and in Chapter II of the work, which is the part that concerns the mathematician, the details of the method are described in general language. The method is well known, and the treatment here does not call for special comment, except that in an interesting section the author discusses the statistical significance of results, firstly by the classical method, going back to Laplace and Gauss, but considered in the final form given it by R. A. Fisher, secondly by the method of Frisch, in particular his "bunch-map" analysis, and finally by the combination of these two methods, considered to be complementary rather than alternative, due to Koopmans. The mathematical details of the calculations required are given in an appendix. J. WISHART.

Coordinate geometry. By L. P. EISENHART. Pp. xi, 298. 13s. 6d. 1939. (Ginn)

We are well accustomed now to look to Princeton for a lead in geometry; and therefore a book on elementary coordinate geometry by Professor Eisenhart, whose treatises on more advanced geometrical topics are classical, is bound to command our attention. In considering its scope, it must be re-

membered that the freshman course at Princeton, for which the book is particularly intended, probably differs in several respects from corresponding courses in this country; any discussion of the book must be subject to recognition of these differences.

There are five main sections: points and lines in the plane; lines and planes in space (this includes an account of determinants); transformations of coordinates; the conics; the quadrics. It is good to have the weight of Professor Eisenhart's authority for the view that, where circumstances permit, the coordinate geometry of three dimensions should be developed with that of two dimensions, and on parallel lines. Thus the treatment of direction cosines is unified, and though this may appear to make the plane work rather more laborious than usual, there is full compensation in the resulting avoidance of all ambiguity. Throughout the book, in fact, the author is concerned to resolve all ambiguities of sign where logic demands such resolution; some wise words deserve quotation: " The question of sign may be annoying, but in many cases it is important; there are also cases when it is not important, and the reader is expected to discriminate between these cases."

The tone of the whole volume is set by the opening section; we do not begin with coordinates, but with a complete account of the possible solutions (x, y) of the linear equation $ax+by+c=0$. One might expect conics and quadrics to be studied as loci defined by equations of the second degree, but actually the conic is defined by the focus and directrix property, and quadrics are introduced after considering the special case of quadrics of revolution. The main properties of these loci, tangents, polars, diameters, are concisely expounded, with due regard to careful reasoning. The distinction between a necessary condition and a sufficient condition is made emphatic; such distinction is too often treated by writers on elementary coordinate geometry as a needless refinement.

Although in the preface the author remarks that " it is not intended that [the reader] should develop no facility in mathematical techniques ", it is in technique that the book seems least successful. Why elaborate the machinery of direction cosines in the plane, if we are to find the locus of mid-points of parallel chords of a conic by substituting $y=mx+h$ and solving the resulting quadratic, especially when the better method is used in the analogous problems for quadrics? Why neglect the elementary but powerful parametric methods, a neglect which entails a rash of repulsive square roots on many pages? The slow-moving student may unhappily be encouraged to regard coordinate geometry as a subject in which " You put down a lot of equations and try to solve them ".

This is undoubtedly a good book for the teacher; it will hardly be a good book for the pupil, unless he is already acquainted with the results, and can concentrate on the trees without losing sight of the wood. There is a short but excellent appendix on the relations of coordinate geometry to the axioms of Euclidean plane geometry. The printing is clear and elegant; the pictures, particularly those of quadrics, are very good. T. A. A. B.

1312. We had heard that critters (yaks) were sometimes inclined to be truculent with Europeans but . . . they were never anything but the gentlest of animals where we were concerned, and more like a collection of absent-minded mathematicians than we would have believed possible.—Ronald Kaulback, *Salween.* [Per Mr. E. G. Phillips.]

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